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ABSTRACT

This collection of articles relating to elementary and middle school mathematics illustrate the goals outlined in the National Council of Teachers of Mathematics (NCTM) documents' six Principles and 10 Standards. These goals provide effective lenses for carefully examining and thinking deeply about the classroom episodes described in each article. The articles illustrate the variety of ways in which the vision of the "Principles and Standards" can be actualized within the constraints, realities, and challenges of today's classrooms. The articles are organized into three parts: (1) Principles; (2) Content Standards; and (3) Process Standards. Individual goals are listed for each principle and standard, and three or four articles are presented as exemplars of each set of goals. Activities are included at the end of each section to encourage further reflection upon the nature of these goals and their implications for classroom teaching and learning. (KHR)

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on NCTM's

Principles and Standards in Elementary and Middle School Mathematics

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NCTM

National Council of Teachers of Mathematics

Reflecting on NCTM's *Principles and Standards* in Elementary and Middle School Mathematics

Readings from NCTM's School-Based Journals

Edited by
Anne R. Teppo
Bozeman, Montana

National Council of Teachers of Mathematics

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INTRODUCTION

Reflecting on NCTM'S *Principles and Standards*

THIS collection of articles, relating to elementary and middle school mathematics, is drawn from the following publications of the National Council of Teachers of Mathematics (NCTM): *Teaching Children Mathematics, Mathematics Teaching in the Middle School*, the *NCTM News Bulletin*, and *Mathematics Education Dialogues*. The collection is intended to promote reflection about the ideas and recommendations put forth in *Principles and Standards for School Mathematics* (NCTM 2000).

The different articles illustrate the goals outlined in the NCTM document's six Principles and ten Standards. In turn, these goals, which reflect the complex, interrelated nature of all aspects of classroom teaching and learning, provide effective lenses for carefully examining and thinking deeply about the classroom episodes described in each article. As a whole, the collection illustrates the variety of ways in which the vision of the Principles and Standards can be actualized within the constraints, realities, and challenges of today's classrooms. Individually, each article opens another window on what outcomes are possible.

The articles are organized into three parts: Principles, Content Standards, and Process Standards. The individual goals are listed for each Principle and Standard, and three or four articles are presented as exemplars of each set of goals. Activities are included at the end of each section to encourage further reflection about the nature of these goals and their implications for classroom teaching and learning.

This book is intended to be a companion to *Reflecting on Practice in Elementary School Mathematics* (Teppo 1999). That book introduces reform-based classroom mathematics, presenting a collection of earlier articles from NCTM publications that "help bridge the gap between readers' past experiences as learners of (often traditional) school mathematics and their roles as teachers within the reform movement" (Teppo 1999, p. 1). *Reflecting on Practice* also includes a brief outline of the NCTM's 1989 Curriculum Standards for grades K–4 and grades 5–8. The present book, while standing alone, carries forward the themes of the earlier collection within the context of the new *Principles and Standards* document.

Principles and Standards for School Mathematics

> Imagine a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction. (NCTM 2000, p. 3)

Principles and Standards for School Mathematics was released by the National Council of Teachers of Mathematics in spring 2000. This book updates and amplifies the vision for improving mathematics curricula, teaching, and assessment outlined by the Council's three earlier Standard documents—*Curriculum and Evaluation Standards for School Mathematics* (1989), *Professional Standards for Teaching Mathematics* (1991), and *Assessment Standards for School Mathematics* (1995).

Input for updating the Standards was collected from many sources including curricular documents, research reports, conferences, and association review groups. A preliminary draft of the new *Principles and Standards* was subjected to an extensive review process by both groups and individuals, and the final document was substantially revised on the basis of the information received. The book presents a vision for the future—

> in which all students have access to rigorous, high-quality mathematics instruction, including four years of high school mathematics, [and in which] the curriculum is mathematically rich, providing students with opportunities to learn important mathematical concepts and procedures with understanding. (NCTM 2000b, p. 1)

Principles and Standards describes a set of six Principles and ten Standards."The Principles should be useful as perspectives on which educators can base decisions that affect school mathematics" (NCTM 2000, p. 7). The Standards present goals for developing the mathematical understanding, knowledge, and skills expected of all students. The first five Standards address mathematical content, and the remaining five address mathematical processes. The ten Standards are also described in greater detail through four gradeband chapters covering prekindergarten through grade 2, grades 3 through 5, grades 6 through 8, and grades 9 through 12. This document is intended to—

- set forth a comprehensive and coherent set of goals for mathematics for all students from prekindergarten through grade 12 that will orient curricular, teaching, and assessment efforts during the next decades;
- serve as a resource for teachers, education leaders, and policymakers to use in examining and improving the quality of mathematics instructional programs;
- guide the development of curriculum frameworks, assessments, and instructional materials;
- stimulate ideas and ongoing conversations at the national, provincial or state, and local levels about how best to help students gain a deep understanding of important mathematics. (NCTM 2000a, p. 6)

The book not only reflects a vision for the future of school mathematics; it is, itself, designed to take advantage of emerging technologies. An electronic version on CD-ROM contains extensive examples that augment the information presented in the hard-copy text. The electronic edition is also available on the World Wide Web at standards.nctm.org.

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Teppo, Anne, ed. *Reflecting on Practice in Elementary School Mathematics.* Reston, Va: National Council of Teachers of Mathematics, 1999.

SUMMARY OF THE PRINCIPLES AND STANDARDS

Principles

1. The Equity Principle

Excellence in mathematics education requires equity—high expectations and strong support for all students.

2. The Curriculum Principle

A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across grades.

3. The Teaching Principle

Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.

4. The Learning Principle

Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.

5. The Assessment Principle

Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.

6. The Technology Principle

Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.

Content Standards

1. Number and Operations

Instructional programs from prekindergarten through grade 12 should enable all students to—

- understand numbers, ways of representing numbers, relationships among numbers, and number systems;
- understand meanings of operations and how they relate to one another;
- compute fluently and make reasonable estimates.

2. Algebra

Instructional programs from prekindergarten through grade 12 should enable all students to—

- understand patterns, relations, and functions;
- represent and analyze mathematical situations and structures using algebraic symbols;
- use mathematical models to represent and understand quantitative relationships;
- analyze change in various contexts.

3. Geometry

Instructional programs from prekindergarten through grade 12 should enable all students to—

- analyze characteristics and properties of two-and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;
- specify locations and describe spatial relationships using coordinate geometry and other representational systems;
- apply transformations and use symmetry to analyze mathematical situations;
- use visualization, spatial reasoning, and geometric modeling to solve problems.

4. Measurement

Instructional programs from prekindergarten through grade 12 should enable all students to—

- understand measurable attributes of objects and the units, systems, and processes of measurement;
- apply appropriate techniques, tools, and formulas to determine measurements.

5. Data Analysis and Probability

Instructional programs from prekindergarten through grade 12 should enable all students to—

- formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them;
- select and use appropriate statistical methods to analyze data;
- develop and evaluate inferences and predictions that are based on data;
- understand and apply basic concepts of probability.

Process Standards

1. Problem Solving

Instructional programs from prekindergarten through grade 12 should enable all students to—

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems.

2. Reasoning and Proof

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

3. Communication

Instructional programs from prekindergarten through grade 12 should enable all students to—

- organize and consolidate their mathematical thinking through communication;
- communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
- analyze and evaluate the mathematical thinking and strategies of others;
- use the language of mathematics to express mathematical ideas precisely.

4. Connections

Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize and use connections among mathematical ideas;
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognize and apply mathematics in contexts outside of mathematics.

5. Representation

Instructional programs from prekindergarten through grade 12 should enable all students to —

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena.

Reprinted from *Principles and Standards for School Mathematics* (Reston, Va.: National Council of Teachers of Mathematics, 2000, pp. 12–67).

PART 1:

PRINCIPLES FOR SCHOOL MATHEMATICS

THE SIX Principles "describe particular features of high-quality mathematics education" and offer guidance for making educational decisions (NCTM 2000, p. 11). The issues addressed by these principles affect all stakeholders in the educational enterprise, including parents, teachers, school administrators and other policymakers, and teacher educators. Although presented as brief statements, each Principle has far-reaching implications for the nature and direction of school mathematics.

> They can influence the development of curriculum frameworks, the selection of curriculum materials, the planning of instructional units or lessons, the design of assessments, the assignment of teachers and students to classes, instructional decisions in the classroom, and the establishment of supportive professional development programs for teachers. (NCTM 2000, p. 12)

In part 1, the articles illustrating each Principle are organized into three groups to emphasize the related nature of particular Principles. Section 1 addresses some of the general educational issues related to the Equity and Curriculum Principles. The Teaching, Learning, and Assessment Principles, grouped in section 2, examine the complex interactions of the mathematics classroom. Section 3 contains articles related to the Technology Principle.

These articles present the collected experience of teachers, mathematics educators, and concerned parents. Their descriptions of teachers and students engaged in mathematical activity provide vivid examples from which the perspectives outlined in each Principle can be examined. Further evidence of the Principles in action can be found in other articles in this volume, and the understandings developed from the readings in part 1 can be used to guide the study of these later articles.

1.

THE EQUITY AND CURRICULUM PRINCIPLES

THE EQUITY and Curriculum Principles are grouped because of the nature of the information included in this section. Taken as a unit, the seven articles look at different aspects of the big picture with which one must come to grips when thinking about reform in school mathematics and where "all students need an education in mathematics that will prepare them for a future of great and continual change" (NCTM 2000, p. 8).

The Equity Principle

EXCELLENCE in mathematics education requires equity—high expectations and strong support for all students.

The equity principle expresses both "an essential goal and a significant challenge" (NCTM 2000, p. 12). The realization of this principle requires—

- high expectations and worthwhile opportunities for all students to learn high-quality mathematics;
- the accommodation of such differences as non-English-speaking students and those with disabilities, special interests, or exceptional talents through solid instructional support, including the innovative use of technology;
- resources and support for all classrooms and all students, including high-quality instructional programs, tools, curriculum materials, and community resources as well as professional development of teachers. (NCTM 2000, pp.12–14)

The four articles included in this group address a range of issues related to the goals stated above. By no means exclusive, these issues illustrate how all-encompassing the phrase *all students* must be.

Zaslavsky defines the terms "ethnomathematics" and "multicultural education." She calls for a restructuring of the educational enterprise to include a curriculum and a set of teaching and assessment practices that are sensitive to all cultures. Following up on the multicultural theme, De La Cruz describes a program designed to encourage Latino parents' involvement with their children's school experiences. A sensitivity to the parents' needs and cultural expectations was essential to the program's success.

Morrow raises the issue of gender equity with calculator use. Because boys have developed a different relationship with technology than have girls, care must be taken to provide a "girl friendly" learning environment. The author describes a summer program in which one of the goals is to help girls become powerful users of technology.

The final article addresses the needs of the non-typical child. Moldavan recounts examples of her young son's mathematical insights. "Matt's" language development had been delayed and he was experiencing great reading difficulties in school, yet his mathematical understanding of out-of school contexts showed ability and innate curiosity. The author uses her story to remind teachers to expect more of their students and to provide a variety of opportunities in which each child can individually excel.

Ethnomathematics and Multicultural Mathematics Education

by Claudia Zaslavsky

For over a decade, teachers, administrators, elders, and other community members of the Yup'ik Eskimo villages in southwest Alaska have been planning ways to integrate traditional Yup'ik mathematics and science into the school curriculum, introducing such topics as the Yup'ik base-twenty system of number words and the symmetry of patterns in clothing and basketry.

A continent away, British mathematics educator Mary Harris assembled a traveling exhibit, Common Threads, to illustrate mathematical concepts intrinsic in women's work with textiles in various cultures. For example, she compares the geometry involved in knitting the heel of a sock—"women's mathematics"—with that of designing a right-angled cylindrical pipe—"real mathematics."

At the same time, sixth graders in a predominantly Latino, low-income community in New York City plan an outdoor play space within their school yard to accommodate the preschoolers to whom they teach mathematics. The sixth graders attend the Community Service Academy of a public middle school, and the preschoolers come from an African American church and a Jewish community center (Zaslavsky 1996).

All these people are engaged in doing ethnomathematics, which is the meeting of cultural anthropology with mathematics and education.

People all over the world are involved in mathematical activities. They count objects, measure various quantities, design buildings and works of art, locate places in space, and play games that incorporate mathematical concepts. Of necessity, they develop terminology to communicate with others about these practices. How they carry out these activities differs from one society to another. We can no longer view mathematics as a culture-free discipline.

The Brazilian philosopher of mathematics Ubiratan D'Ambrosio introduced the term *ethnomathematics* in the 1970s in speeches emphasizing the influence of sociocultural factors on the teaching and learning of mathematics. In his usage, the prefix "ethno" encompasses identifiable groups, such as national societies, ethnic groups, works in a specific field, professional classes, local communities, or children of a certain age, and includes their jargon, codes, symbols, myths, and ways of reasoning. By no means is a precise definition of ethnomathematics accepted by all who are interested in sociocultural perspectives in mathematics, nor is complete agreement found about the term that should be used to describe such perspectives. In *Africa Counts: Number and Pattern in African Culture* (1973), I wrote about the sociomathematics of African peoples. Ethnomathematics is not confined to ethnic mathematics; it includes the mathematics developed by various ethnic groups but has a wider scope as well.

Although ethnomathematics is concerned with the mathematical experiences of various groups, multicultural education is a national movement that encompasses all aspects of education for all students at all grade levels and involves all disciplines. Ideally, multicultural education is a program that promotes the right of all students, whatever their gender, ethnic and racial heritage, or socioeconomic status, to receive the highest possible level of education that will enable them to understand the issues and problems of our diverse society and later solve them. Students should be encouraged to develop skills in critical thinking and analysis that can be applied to all areas of life, including vital issues involving race, gender, and socioeconomic class.

Claudia Zaslavsky is currently retired after many years of teaching secondary-level mathematics and teacher-education courses. She is a former secretary of the International Study Group on Ethnomathematics and is the author of adult and children's books dealing with multicultural perspectives in mathematics.

Multicultural mathematics education requires the inclusion in the curriculum of the ethnomathematics of all peoples—ethnic and racial groups, women and men, the various classes in society—as well as the practices and problems of the students' own communities. Although school mathematics usually emphasizes European contributions, much of the mathematics in the elementary- and middle-grades curriculum originated in Africa and Asia (Joseph 1991). Students learn that real people in all parts of the world and in all eras of history developed mathematical ideas because they needed to explain their environment and to solve the vital problems of their daily existence. Unfortunately, schools tend to ignore the ethnomathematical knowledge that the children themselves can bring into the classroom from their homes and communities. When mathematics is placed in a relevant context, it has meaning for students.

Multicultural perspectives provide a link among mathematics and other subjects in the curriculum, such as social studies, language arts, science, and fine arts. Multicultural mathematics projects afford opportunities for cooperative learning and for students of diverse backgrounds, interests, learning styles, and achievement levels to work together toward shared goals. They can take pride in the contributions of their own cultures and learn to appreciate the achievements of other cultures.

Multicultural mathematics education deals with both the content and the process of education—curriculum, classroom management, teacher expectations, and professional development—and relations among teachers, administrators, students, and the community. I envision a restructuring of the educational enterprise to encompass the following:

- Mathematical concepts and ethnomathematical perspectives must be synthesized in the curriculum in a way that is sensitive to all cultures. New textbooks and teaching materials must be produced with this synthesis in mind.
- Teachers must be well grounded in mathematics and at the same time become familiar with the interface among mathematics and other subject areas. They will need the time to work with other teachers, parents, and the community in planning lessons that are relevant to their students.
- Assessment instruments must be appropriate to the curriculum. Simplistic multiple-choice tests must be downplayed or abolished.
- Educators must employ the practices that will best enable underserved and underrepresented groups, particularly females and people of color, to learn mathematics. Tracking by so-called ability levels must be abandoned.

The incorporation of ethnomathematical perspectives calls for a complete turnaround from the way that mathematics is now taught in many classrooms. Carrying out a change of such magnitude is a slow process, so let's get started!

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Reversing the Trend: Latino Families in Real Partnerships with Schools

by Yolanda De La Cruz

Señor Lopez walked up to the project director after the last mathematics class and said, "De todos los años que he tenido hijos en estas escuelas, esta es la primera vez que me han tratado con tanta dignidad" (In all the years I have had children attend these schools, this is the first time that I have been treated with so much dignity).

Involving parents in their children's learning has been recognized as necessary and important by scholars who contributed to several reports on the nation's schools (De La Cruz 1999; Epstein and Becker 1982; Epstein 1983). The recommendation reflects the consistent findings in social science research that children have an added advantage in school when their parents encourage and support schooling.

Latino families are growing in numbers; currently they compose the largest minority group in the Southwest and have been visibly transforming culture and society (Takaki 1996). These families are seeking ways to improve their status in the United States and look to the educational process as a way of achieving this goal (Khisty and Becker 1990; Secada and De La Cruz 1996). By including them in real partnerships with schools, we can help them reach these goals.

This article outlines a research-based model for greater family involvement among Latino families in the area of mathematics. The Children's Math Worlds Family Connection (CMWFC) is a component of a reform-mathematics curriculum called Children's Math Worlds (CMW), which was developed for first through third grades under the leadership of Karen C. Fuson at Northwestern University. The curriculum was developed over the past seven years, initially in urban Latino classrooms, but more recently in a range of other classrooms. The CMWFC component was directed by Yolanda De La Cruz at Arizona State University West and has been under development for three years.

Formal assessments were conducted of the mathematical understandings of various classes of urban children learning from our pedagogy (Fuson, Smith, and Lo Cicero 1997; Fuson 1996; Fuson 1998). Although more than 90 percent of our urban CMW children met federal guidelines for the free-lunch program, they considerably outperformed heterogeneous and middle-class samples of United States children who received traditional mathematics instruction. On many items, they outperformed children from Taiwan and United States children using the reform curriculum Everyday Mathematics (UCSMP 1998); and on some tasks, they equaled or exceeded the performance of Japanese children. On standardized tests, 90 percent of the urban children scored at grade level on computation and 65 percent on problem solving. Class means on overall mathematics scores were above grade level, some children scored three years above grade level, and no child scored more than one year below grade level. The results for suburban children were even stronger. The usual suburban-urban gap still existed but was smaller on some items.

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Factors Affecting Latino Children's School Achievement

Several factors affect Latino children's school achievement (fig. 1). Often, traditional curricula do not offer enough steps to help students with limited-English-speaking ability build better conceptual understanding. These students, therefore, fall farther and farther behind, leaving many of them with large knowledge gaps.

As more school districts require teachers to implement reform mathematics curricula, little support is offered in helping them develop strategies for new demands placed on their teaching. For example, teachers are finding that reform-based mathematics instruction places more demands on facility with oral and written English. They do not have the strategies, however, that enable them to work more effectively with their limited-English-language students.

Many parents of Latino students either speak no English or are limited-English speakers; thus, communication between parents and teachers is not simple. Teachers express deep concern that not enough family participation occurs among Latino families, but they do not know how to develop partnerships with these families. More support is needed to help teachers overcome barriers that prevent them from building partnerships with the families of their students.

Four areas affect the mathematics learning of Latino students and prevent many of them from gaining access to this knowledge.

- Traditional curricula often leave gaps in conceptual understanding.
- Reform-based mathematics instruction requires facility with oral and written English.
- Communication with limited-English-speaking parents is difficult.
- School programs often fail to include families in a teaching partnership.

Fig. 2.1

True partnerships with Latino families will result when more emphasis is placed on ascertaining their needs and convenient times for them to attend programs to acquire resources to help their children. Communication and language barriers must be overcome before any true partnership can result. Families must be treated with the dignity that they deserve, and both teachers and parents must make an effort to develop and nurture partnerships that can reverse the trend of Latino underachievement in our schools.

CMWFC Approaches to These Issues

School reform programs are often developed without sufficient attention to how to attain the academic goals required for success among Latino and other linguistic-minority and ethnic-minority students. Several approaches helped to make the CMWFC successful in meeting the needs of Latino families. In the CMWFC, we asked parents what they needed to be able to help their children in mathematics. Print communication was always done in both Spanish and English. We sent surveys home with children and made follow-up calls to parents who had not responded to the survey. We found that parents wanted to learn to help their children at home but did not think that they had enough mathematics knowledge to help their children in grades 1 through 3.

We sent activity booklets home to families for each grade level, first through third, containing activities and games that reinforce school learning. The booklets were available to families throughout the school year. Figure 2.2 and figure 2.3 are examples of some of the information contained in these booklets. These games and activities reinforced the mathematics curriculum being taught in the school.

Families were invited to attend two workshops during the school year to help them improve their mathematics knowledge. These sessions were scheduled for parents with or without their children in attendance. In Illinois, we found that families preferred to bring their children to these workshops. The classes were given for parents first while children attended a separate program. After the first hour, the children joined their parents for the remainder of the workshop. These parents remarked that they felt more confident once they had successfully done the activities without their children. We gave the mathematics workshops at times that were best for parents, as determined by the survey responses. We found that in both Illinois and Arizona, families preferred late-afternoon and week-end mathematics workshops. In Illinois, one hundred families attended the mathematics classes with their children, even in –43-degree-wind-chill weather.

Families viewed pilot videotapes, available in both English and Spanish, showing them how to do the mathematics activities. Families who attended the workshops checked out the videotapes from the school library whenever they wanted to review how to do the activities.

CMWFC offered workshops for teachers interested in gaining more family support. Teachers were given information on how to respect the needs of parents and how to treat them with the dignity they deserve. Eleven teachers attended the classes because their prior efforts had failed to produce family involvement. They found that sending letters home in both Spanish and English and making follow-up telephone calls made a great difference in gaining family support. When classes were scheduled at times that parents

How Many Pennies Did I Hide?

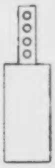
Materials:
1 penny strip

Players:
2 players

Purpose/mathematics skills practiced:
To learn which pairs of numbers make 10

Rules of the game:
A penny strip is laid penny side up in front of the two players. One player covers a number of pennies in the penny strip with a hand or a piece of cardboard and asks his or her partner, "How many pennis did I hide?" The partner has to figure out how many were covered and then checks the answer by uncovering the hidden pennies. If the partner gets it right, he or she gets a point.

Children take turns being the one to hide the pennies in the penny strip.



How many pennies did I hide?

Harder variation: When your child is good at this game, she or he can figure out how many pennies are covered. In this version, your child cannot see the penny strip. She or he has to do it with eyes closed! Ask, "I have four pennies left. How many did I hide?" or "I hid six pennies. How many did I had left?"

This game will help your child learn combinations of 10.

Fig. 2.2

¿Cuántos Centavos Escondí?

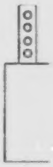
Materiales:
1 tira de centavos

Jugadores:
2 jugadores

Objetivo/Habilidades de matemáticas que se necesitan:
Aprender cuáles pares de numeros hacen 10

Reglas del juego:
Una tira de centavos se coloca con los centavos boca arriba frente a dos jugadores. Un jugador cubre un número de centavos en la tira de centavos (con la mano o una hoja de cartón), y le pregunta a su compañero "¿Cuántos centavos escondí?" El compañero tiene que comprender cuántos estaban escondidos y luego comprueba la respuesta descubriendo los centavos cubiertos. Si tiene razón, recibe un punto.

Los niños se turnan para ser el que cubre los centavos en la tira de centavos.



¿Cuántos centavos escondí?

Variación mas difícil: Cuando su niño hace bien este juego, puede comprender cuántos centavos están cubiertos. No puede ver la tira de centavos. ¡Tiene que hacerlo con los ojos cerrados! Pregunte Ud, "Me quedan 4 centavos. ¿Cuántos cubrí? o "Cubri 6 centavos. ¿Cuántos me quedaron?"

Esto ayudará a su niño a aprender las combinaciones de 10.

Fig. 2.3

had requested, teachers were pleasantly surprised by the overwhelming parental response. Parents remarked that they had never been given a choice of times for school-related activities. They felt like real partners in the learning and teaching of their children.

Participants' Responses

Parents were asked to give written or oral evaluations after the workshops. The following outcomes are indicative of their responses.

- One father worked at night, and he was accustomed to sleeping in the late afternoon. But he decided instead to attend with his wife the mathematics classes offered through CMWFC so that they could both help their children with homework.
- One mother said that in Mexico, teaching was customarily left for the teacher. Here in the United States, she learned that teachers expect parents' help with homework.
- Another couple attended because they liked the games that their child was bringing home and wanted to learn how to play them.
- Many parents appreciated that only inexpensive materials were needed to play the games and activities.

Teachers were asked whether they noticed any differences due to the CMWFC workshops. Teachers replied that some of the parents approached them for various reasons related to mathematics homework, something that had not happened in the past. Typically, many Latino parents would not think of contacting their children's teachers, but because these parents participated in CMWFC, they felt comfortable enough to do so.

Future Directions

We are in the process of developing a series of home videotapes showing parents how to play various activities and games with their children. More games and activities are being developed to reinforce school mathematics learning in the home. Teachers are making their own videotapes to address particular needs of their Latino students' parents. They tape students as they explain a method or procedure so that parents gain more effective ways of helping their children. These videotapes explain to parents how to focus on the process instead of just on getting the correct answer. They give guidelines and sample questions that parents can use for this process.

Various community organizations have expressed an interest in expanding the program. Parents will be trained to help facilitate workshops for families. Teachers who are currently involved will train parents for future CMWFC workshops. Parent and teacher manuals for conducting workshops are under development. These manuals will describe how to meet teachers' and parents' expectations.

We are confident that these efforts to expand the program will help more schools and Latino families develop true partnerships.

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Ensuring That All Children Are Powerful Technology Users

by Charlene Morrow

A CALCULATOR is not only an essential tool in today's world but a bridge for young children into the world of scientific technology. Our reluctance, as educators, to teach children to use calculators as a mathematical tool promotes inequities that will later affect students' access to college courses and to work. When we don't structure equitable opportunities for children to use technology, we allow social forces to take over, leading to more comfort with technology on the part of, and usage by, males than females and more comfort on the part of those students whose families are willing and able to buy and use technology.

Historically, the use of tools has empowered tool users to gain control over their environment and to become experts, with associated economic benefits. In fact, the use of tools, along with language, is the essence of being human. The use of technologically advanced tools, especially those associated with paid work rather than housework, has been the province of males. We will need to work hard to make sure that these cultural factors do not prevent females (and other historically disenfranchised groups) from having equal access to important modern-day tools. The American Association of University Women (AAUW), in its recent publication Gender Gaps: Where Schools Still Fail Our Children, has identified the use of technology as having among the most significant gender gaps of any area in school.

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Lest you are about to jump up and toss calculators into the classroom, let me caution you that this is where things get tricky. What if calculator use does not promote a "girl friendly" learning environment? According to the AAUW again, "Girls have developed an appreciably different relationship to technology than boys...and technology may exacerbate rather than diminish inequities by gender as it becomes more integral to the K–12 classroom." More boys have and use technological tools and toys. Boys more often perceive themselves as going into careers, such as engineering, that require technology like calculators and computers. They can envision a payoff for learning to use these tools. We need to be sensitive to past and present inequities when structuring opportunities for all children.

In our own program, SummerMath, a four-week program for a diverse group of high school girls of varied mathematical experiences, girls enthusiastically take to technology—both calculators and computers. But we are not satisfied with girls using technology only when others say to do so. Our objective is for girls

to be powerful users of technology—initiators of technology use. This summer we are recognizing the importance of the ownership of tools by buying each student a graphing calculator to take home.

Unstructured experiences with technology will simply preserve the social order—opportunities need to be structured in order to be equitable. The AAUW has many suggestions on this topic. It is incumbent on us as educators to ensure that these structured opportunities occur in school. Although mathematics educators argue about whether or not to teach and encourage children to use calculators from an early age, some children—-more males than females—will be learning to use them anyway and thus will be ready for a wider array of college and career opportunities.

Observing the Power of Matt, the Mathematician

By Carla C. Moldavan

As the National Council of Teachers of Mathematics seeks to enhance connections among home, community, and education, perhaps more attention should be paid to the contributions that parents can make to demonstrate the growth of students' mathematical power. In this article, I open a public window on my own son's mathematical thinking processes (NCTM 1995, p. 58). After relating some of my treasured episodes, I discuss implications for teachers.

My decision to begin a "parental portfolio" was simply a matter of not wanting to forget some of the mathematical thinking that Matt had shared with me. Most of these events have never been recounted to Matt's classroom teachers. As a mathematics teacher myself, however, I see potential value in preserving these vignettes for a wider audience of teachers.

Mathematical Memories

Matt's language development was delayed, and he qualified for preschool speech therapy through the local schools when he was three years old. A few months before his third birthday, Matt had only about twenty words in his vocabulary, including some that he had used for a while but was no longer using. The first book that he truly enjoyed having me read to him was *Little Toy Train.* He pointed to the cars with his fingers to count them and seemed to understand the concept of one-to-one correspondence long before the words for numerals were in his vocabulary. Matt's attempt to count the train cars at age two went like this: "hee-hee, ha-ha, ho-ho." A year later, he had progressed to "one, two, three, M, N, O."

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In kindergarten and first grade, Matt asked me such questions as "How many stories high does an airplane fly?" and "How many houses will an airplane hold?" Possibly because I had told the first-grade teacher that Matt raised questions like these, she called him "Matt, the Mathematician." The nickname may also have sprung from the fact that she always looked for the strengths of every child and made learning a positive experience. The teacher was even thoughtful enough not to ask Matt questions that would create opportunities for other children to make fun of the way that he said *two.*

Matt's spontaneous sharing during second grade caught me by surprise many times. One day he reported to me that his class had walked over to the high school track for physical education. He told me that it took four laps to make one mile. I was delighted as I watched his eyes and gestures while he was thinking. Then he explained to me why each lap would be "two-and-a-half tenths" of a mile. I assume that he thought in tenths because he was familiar with the car's odometer.

One day during the summer after second grade, Matt and I were talking about the fact that an electrician would be coming to our house to do some work. Matt asked how much the service would cost. I told him that the charge would be $25.00 for the service call, plus the cost of the parts; thus, I estimated that the total would be between $50.00 and $75.00. Matt's reply was that I could just say $62.50 because that is in the middle. To calculate his "average," though he did not call it that, Matt determined that $60.00 would be the average of $50.00 and $70.00, then split the other $5.00 into two groups of $2.50.

A few days later, Matt went with me to buy some new living room furniture at a store. When the saleswoman sat down at her table with a calculator to figure the tax, I asked Matt to figure it mentally. Knowing that the tax was $6.00 for every $100.00, he reasoned that for $400.00, the tax would be $24.00, plus another $3.00 for the other $50.00; therefore, the tax on $450.00 would be $27.00. I asked him to tell me the total amount to write he check for. He wanted to be lazy, but the saleswoman encouraged him to confirm her calculations.

For Christmas when Matt was in third grade, he got a remote-control car. He read the instructions to determine how long it would take to fully charge the battery and how long the car would run on a full charge. Then he announced to me that each hour of recharging would yield about five minutes of running time.

Another toy prompted Matt to do some more mathematical thinking. Looking at his Lego catalog, Matt questioned why the price of Lego sets was not proportional to the number of pieces in the set.

Sometimes Matt's insights came while he was riding in the car with me. One day, he estimated that the huge flag he saw as we were riding down the road was about two car lengths, or one truck length, long. Another day, Matt informed me that one side of the road was longer than the other. At first I did not understand this observation. Matt's explanation showed that he was conceptualizing that on a curved section of road, the perimeter of the inner part of the curve would be less than the perimeter of the outer section of the curve.

Sometimes our mathematical interactions represented teachable moments in which I could facilitate a meaningful exploration. One day Matt was playing with my scientific calculator and entered a multiplication problem for which the product was expressed in scientific notation. After hearing my brief explanation of scientific notation, he began entering a multiplication problem with many factors, each of which had several zeros for digits. Expecting an extremely large number, he blinked in disbelief as he saw 0 displayed for an answer. I helped him diagnose his "problem" by asking what would have to be true about one of the numbers, if he had multiplied a group of numbers together and got 0 for the answer. I had to be more direct, asking if I multiplied a group of numbers together, none of which was 0, could I possibly get an answer of 0? Then we looked on the display and found that he had indeed accidentally entered 0 as a factor.

The experiences of watching my son's growing mathematical power has been enriching, and I am thankful for it. In other areas, I have seen my son have great difficulty. I thought that Matt would never recognize the word *this*, even after reading it for what seemed like a million times. Although we certainly could have won the award for reading more than any other first grader in the school, reading and written expression did not come easily for Matt. He can draw and build with Legos, demonstrating phenomenal spatial visualization, but his handwriting is still not up to expectations. I offer these comments to show that parents can give a balanced view of their children, but I am still proud of Matt's curiosity about, and abilities in, mathematics.

Implications for Teachers

What messages do these episodes convey for teachers?

1. Ask parents whether they have observed their children applying skills in everyday settings. Even I am hesitant to share Matt's vignettes with his teachers, but if I were asked during a parent conference, I would candidly give examples.

2. Use examples of everyday settings that parents might incorporate in daily interactions with their children. Parents can benefit from guidance in realizing the opportunities that they have for creating mathematical moments.

3. Envision your students as capable of much more than you ever dreamed. Do not box yourself into a curriculum of addition and subtraction worksheets when your students could be calculating the tax on major purchases. Expect to be surprised by your students' mathematical gems from time to time. Expect them to be problem solvers.

4. Listen carefully when your students are explaining their observations. Just as I was not sure what Matt meant when he said that one side of the road was longer than the other, so do teachers sometimes dismiss unique ideas as "wrong." A little discussion may uncover a mathematical gem. Communication is prized in a Standards-based classroom.

5. Be grateful for inquiring minds. You will not always be able to answer such questions as, how many houses can an airplane hold? but you can respond in a way that acknowledges an earnest desire for students to both ask and answer questions.

6. Allow assessment to be "a convergence of evidence from a variety of sources" (NCTM 1995, 19). You might have a student with a score in the ninetieth percentile for computation on the Iowa Test of Basic Skills but a fourth-percentile score for mathematical concepts on the same test. When the evidence does not converge, when evaluation is not clear, seek other avenues of interpretation. The concern here is not for a mark to go on a report card but rather for a meaningful picture of a child's development.

These suggestions may prompt you to see other lessons from the experiences of just this one child. If you like, I could tell you about my other son. Noah is seventeen . . . .

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The Curriculum Principle

A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across grades.

"A school mathematics curriculum is a strong determinant of what students have an opportunity to learn and what they do learn" (NCTM 2000, p. 14). Such a curriculum should—

- organize and integrate important mathematical ideas, help students build on and make connections among these ideas, and sequence lessons coherently across school years;
- include emphasis on foundational ideas, skills, and processes that enable students to understand other mathematical ideas, develop new insights, and promote further study;
- focus on mathematics content and processes that are worth the time and attention of students, that support the development of quantitative literacy, and that are relevant and useful in a technological world;
- be clearly articulated to provide guidance about the depth of study warranted at particular times and the location of points of closure for particular skills and concepts across grade levels. (NCTM 2000, pp.14–16)

The three articles in this group present a variety of broad perspectives to reflect on as we try to articulate general directions for the school mathematics curriculum. In the first article, Rock and Shaw discuss an Internet survey of kindergartners' through eighth graders' perceptions of mathematicians and the work that they do. The authors found that younger students tended to draw pictures of people smiling while doing mathematics. However, most of the responses showed these individuals engaged in "school mathematics" activities. The authors stress the crucial role that school experiences play in children's valuations of mathematics and their perceptions of the discipline's effect on their lives.

In the second article, Shirley overviews the many changes that took place in mathematics during the twentieth century. The author summarizes key events by decade in a table listing achievements in applied mathematics, computers, and pure mathematics, as well as in education and society. Although Shirley suggests the study of the field's recent advances is a way to help students see mathematics as a dynamic and changing field, this study is also important when considering what areas of the discipline are appropriate for the school mathematics curriculum for the twenty-first century.

In the final article, Ritchhart illustrates the notion of a generative topic using a vignette of first- and second-grade teachers engaged in redesigning a measurement unit. The author defines a generative topic as being concerned with core ideas and as building understanding through a focus on mathematical entities central to these ideas. The vignette shows how the teachers' deliberations clarified and developed their own understandings of the complexity and connections involved in measurement. The article includes descriptions of the classroom implementation of the redesigned unit to illustrate the power of a generative topic in realizing the vision put forth in the NCTM's *Standards* documents.

Exploring Children's Thinking about Mathematics and Their Work

by David Rock and Jean M. Shaw

MATHEMATICIANS work on exciting projects, such as developing the new international space station, the space shuttle, cellular telephones, ATM machines, and even amusement-park rides. They guide businesses in conducting financial transactions. The work of mathematicians touches us every day. As we have learned from such reports as *A Nation at Risk* (National Commission on Excellence in Education 1983) and *Everybody Counts* (National Research Council 1989), however, fewer and fewer students go on to careers that use mathematics. Probing children's thinking about mathematicians and their contributions and roles in society fosters insights for classroom teachers. On the basis of what children think about mathematicians, teachers might alleviate misconceptions and facilitate and broaden children's thinking about their roles as future mathematicians.

We were familiar with a number of studies concerning children's thinking about scientists, including who they are and what they do. Several educators have used the "draw a scientist" task with elementary school children. They have found, in general, that children portray scientists as white males working in laboratories, often sporting unruly hair, glasses, and lab coats. Children draw and write about scientists working alone, and they comment that scientists work to benefit others through inventing things and making discoveries. Frequently, children picture scientists in well-equipped laboratory facilities using such tools as lenses, microscopes, beakers, and test tubes (Fort and Varney 1989; Krause 1977). Barman's (1999) study of students' thinking about scientists showed similar results. In addition, Barman noted low percentages of students who represented scientists using examples of technology, such as computers and calculators, in their drawings.

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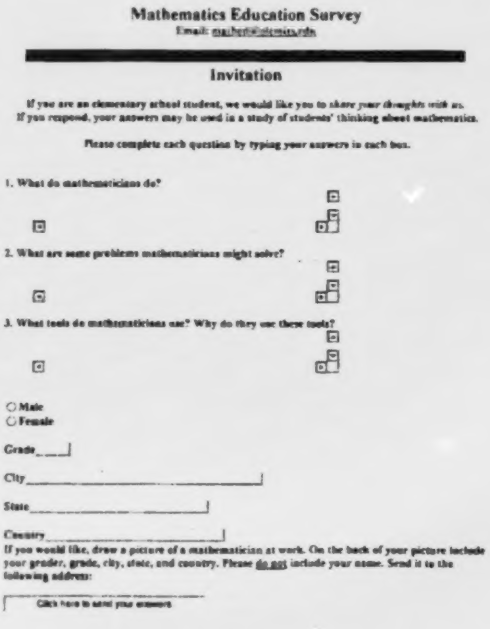
What Do Children Think about Mathematicians?

After reviewing the results of these studies, we became curious. We asked ourselves, "What do children think about mathematicians?" To answer this query, we came up with three questions: (1) What do mathematicians do? (2) What types of problems do mathematicians solve? (3) What tools do mathematicians use? These three questions were used to create a voluntary survey (shown in fig. 5.1) for students in kindergarten through eighth grade. This survey was placed on the Internet and linked to a pair of mathematical problem



Mathematics Education Survey

Email: [illegible]

Invitation

If you are an elementary school student, we would like you to *share your thoughts with us.*
If you respond, your answers may be used in a study of students' thinking about mathematics.

Please complete each question by typing your answers in each box.

1. What do mathematicians do?

2. What are some problems mathematicians might solve?

3. What tools do mathematicians use? Why do they use these tools?

○ Male
○ Female

Grade____

City____________________

State________________

Country________________

If you would like, draw a picture of a mathematician at work. On the back of your picture include your gender, grade, city, state, and country. Please do not include your name. Send it to the following address:

Fig. 5.1

contest sites. Each contest site asked students to voluntarily share their thoughts about mathematicians by clicking on a link designed to take them to the questions. We also invited children to mail drawings of mathematicians at work. The completed surveys were anonymously e-mailed to us. The survey continued for two weeks.

Results

We received responses from 215 children. Table 1 shows the number of respondents from each grade and the number of male and female respondents. The following sections present the results of our analysis of the responses and discussions of children's thinking about mathematics.

Mathematicians at Work

What do mathematicians do? We found that children tend to think that mathematicians do the same kind of mathematics that they themselves do in the classroom. The most frequent response by kindergartners to the question "What do mathematicians do?" was "They do the calendar." Kindergartners also frequently responded that mathematicians "teach numbers" or "learn the kids to do their numbers." Responses focused on numbers and included "counting," "work with numbers," and "make numbers." One of the very few negative responses we received was from a kindergartner who said, "Mathematicians are monsters, and they are about to eat these people."

First graders were more general in their descriptions of mathematicians' work. From such responses as "Do math all day. Maybe he will come home in the night" and "They work very hard," we inferred that the younger students seem to think that mathematicians are both smart and diligent. Other positively worded responses were "They do problems, I think they are great!" "A mathematician is good in mathematics," and "A person who knows math and can do math good."

Table 5.1
Survey Responses by Grade and Gender

Grade	Number of Respondents	Female Respondents	Male Respondents
Kindergarten	22	12	10
First	71	34	37
Second	24	11	13
Third	39	19	20
Fourth	18	10	8
Fifth	1	0	1
Sixth	6	1	5
Seventh	26	11	15
Eighth	8	5	3
Total	215	103	112

About one-fourth of the kindergarten and first-grade children portrayed mathematicians as people engaged in such activities as cutting the grass, painting pictures, and living in attractive homes. These responses led us to believe that these students did not fully comprehend the nature of the question. It is possible that some of these children picture mathematicians in the same way that they do their parents and believe that the work of a mathematician is the same as that of any other adult.

Second and third graders expanded the scope of their comments and drawings to the workplace. About sixty-two percent of second-grade respondents showed and described mathematicians working in such places as a bank, grocery store, and doctor's office (see fig. 5.2). Other second graders characterized mathematicians as working in classroom environments. The majority of third graders also depicted mathematicians in the classroom (fig.5. 3) and tended to picture people working with basic operations on the chalkboard or with pencil and paper.

About a third of our fourth-grade respondents stated that mathematicians teach mathematics to others and "have them do their problems." Others wrote that mathematicians "make up problems and solve them" and "make up things that have to do with math." Our single fifth-grade respondent stated, "Mathematicians solve math problems."

When looking at sixth, seventh, and eighth graders' responses, we found similar patterns. The majority of the students believed that mathematicians simply "did math" or "solved math problems." The general opinion was "Math is what mathematicians do; no more, no less." Not one respondent submitted a real-life example or explanation of what a mathematician might do outside the walls of a classroom. Their input was quite vague, which seemed to indicate that middle-grades students share a lack of knowledge about the types of work that mathematicians encounter in the real world.

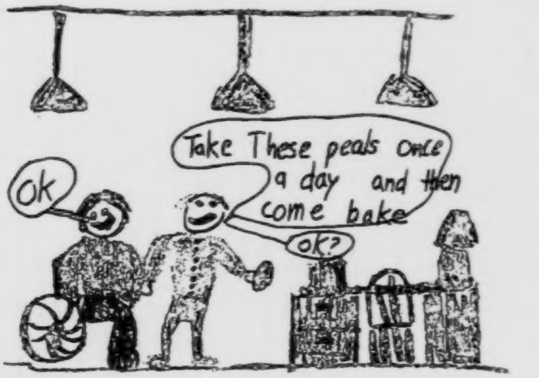

Ok
Take These peals once a day and then come bake
ok?

Fig. 5.2. A second-grade boy shows a doctor as a mathematician. He also wrote, " My dad is a doctor he uses math with patients and medicine."

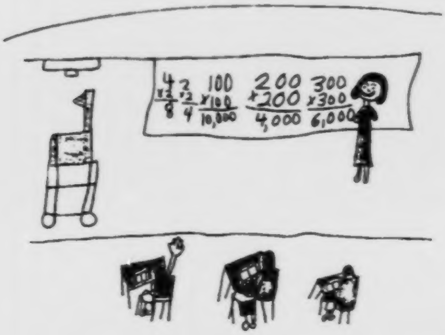


Fig. 5.3. A common response from second and third graders was to show mathematicians at work in their classroom. A third-grade girl sent this picture.

Problems Mathematicians Solve

What kind of problems do students think mathematicians do? Typical of responses made by almost three-fourths of the children was "hard problems other people don't know." Children in kindergarten, first, and second grades were not specific in their comments; almost all made such responses as "They do hard problems" and "They count things."

Third and fourth graders pictured the "hard problems" that mathematicians solve with notations, such as "1,000,000 × 1,000,000,000" and "184 + 674." Four children stated that mathematicians "figure out new problems," and one responded, "A mathematician is being a genius." Six gave responses that involved using applied mathematics, stating that mathematicians "do things like going into outer space" or "work with money" and "use math with patients and medicine." Only one child alluded to the spatial side of mathematics, commenting that mathematicians "spend most of their time solving problems with shapes and designing things like geodesic domes."

About 23 percent of the middle school participants' responses to this question centered on the idea that mathematicians did "hard" and "complicated problems," as well as "problems that no one else could solve." Other students stated that mathematicians were needed to "solve very difficult problems" or "problems that take a long time." Mathematicians are also needed in such endeavors as "finding how long it would take to get to the sun" or determining the "distance from the moon to one of Jupiter's moons."

From these responses, we concluded that students believe that a mathematician's job is to do the mathematics that no one else wants to do. It seems that respondents might even believe that the world has mathematicians solely because someone is needed to do the "hard" mathematics that no one else can do. Some of our respondents also associated mathematicians with extremely large numbers; perhaps they link large numbers with "hard mathematics."

Mathematicians' Tools

What tools do mathematicians use? We wondered whether children would envision mathematicians using tools other than the ones they use in the classroom. Kindergarten and first-grade respondents named pencils, paper, chalkboards, calendars, and number cards—tools with which they are familiar from their own classrooms.

Second- and third-grade children described mathematicians as frequently using both calculators and pencil and paper in their problem solving; almost equal numbers of these responses were received. About a dozen children named erasers "to get rid of your mistakes." One young child stated that mathematicians used "brain, mind, and hands," and another child, that they used "flashcards, fingers, and blocks so they can count and figure out the right answer." Rulers, geometric shapes, templates, typewriters, and machines were also mentioned by a few of the children as tools of mathematicians.

Fourth graders expanded their responses to include computers and calculators. About a third of these students mentioned calculators as valuable "because they help you work out the numbers." Children continued to name pencil and paper as tools of the mathematician. Our fifth-grade respondent named four tools: computers "to speed up calculations," pencil and paper "to work out smaller problems and write down problems and ideas," the brain because "without it the problems would be useless," and calculators.

About 58 percent of our middle school students named calculators as the number-one tool used by mathematicians. Surprisingly, the brain and the protractor tied for second place in our responses. The protractor may have appeared toward the top of the list because many middle school students use a protractor in their mathematics curricula.

One of the most positive results of the survey was to see the large number of both younger and older students who realize that the brain is a significant tool in mathematics. Perhaps they recognize that the brain extends work with manipulatives, shedding insights into problem-solving procedures and guiding work with other tools, such as calculators and computers.

Drawings of Mathematicians

We received 132 drawings from kindergarten through fourth-grade students. Kindergartners often included numbers in their pictures, and first graders usually showed basic facts in their drawings of mathematicians (see fig. 5.4). Virtually all the young children's pictures showed classroom scenes, often with displays of numbers, calendars, number lines, and such tools as pencils, papers, chalkboards, and erasers. Most of the children's drawings showed smiling figures and more female than male figures; more than half indicated no definite racial characteristics. When racial characteristics were shown, we found that three-fourths of the children showed Caucasians and approximately one-fourth showed African Americans.

Second, third, and fourth graders drew mathematicians as males and females in almost equal numbers. More than half the mathematicians were shown in classrooms, most commonly working with other people. More than 80 percent of the figures drawn were smiling while working. The remainder of the drawings showed figures that appeared to be serious or intense rather than sad or depressed. One child drew a "mad scientist" in a science laboratory with $E = mc^2$ written on a chalkboard.

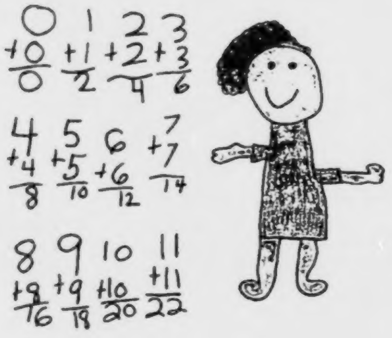


Fig.5.4. A first grade girl's vision of a mathematician includes written work with doubles.

We were extremely encouraged to see that the younger students portrayed people smiling while doing mathematics. This depiction confirms the belief that younger children enjoy doing mathematics; perhaps they think that others have this same positive attitude. We must learn to foster and nurture this excitement and enjoyment through the upper elementary and middle school years and guide students to see that even the hard work that is sometimes required in doing mathematics is worthwhile and satisfying.

Reflections

When we teach in the classroom, it is extremely important to stress that mathematics is used in everyday life—by mathematicians and others. We must make connections to children's everyday lives to help them learn to value mathematics. If we want more students to pursue mathematics as they continue their educations, we must continually emphasize how mathematics is used in the real world.

Children also need to learn about mathematicians in history and mathematics as a modern-day career choice. The Standards state, "Understanding of mathematical ideas should grow and deepen across the grades" (NCTM 1998, p. 33), and we believe that this process involves extended awareness of the importance of mathematicians in society. Through books, Web sites, people in the community, and other resources, educators can help children explore the achievement and work of mathematicians. To guide children's thinking, teachers might share the idea that the study of mathematics was a hobby for some mathematicians. For others, expertise in mathematics led to personal gain. In addition, many mathematicians work hard on the subject because they find the work enjoyable and personally challenging.

Resources

Several resources can help spark students' fascination with mathematics. One way to hook students' interest in mathematicians is to read aloud stories, such as those in volumes 1 and 2 of *Mathematicians Are People, Too* (Reimer and Reimer 1990, 1995), collections of intriguing anecdotes about famous mathematicians. Some of the many other books on mathematicians include *The Librarian Who Measured the Earth* (Lasky 1994), *Classic Math* (Johnson 1994), *Agnesi to Zeno* (Sanderson 1996), and *Celebrating Women in Mathematics and Science* (Cooney 1996).

People in the community can also serve as resources to help students clarify ideas about mathematicians' work. Mathematics majors from colleges might visit the classroom to describe their studies and share their hopes for jobs after graduation. Bankers, statisticians, or engineers might also be invited to make presentations about their work, their tools, and the many ways that they use mathematics. Videotaping such presentations will allow this information to be shared again and again. Teachers may want to appoint students to act as hosts and moderators for these visits; these children can present questions prepared by the class.

Finally, the Internet is a powerful, accessible tool for helping children develop ideas about mathematicians and their work. Some Web sites, current as of this writing, include the following:

- "Ancient Geometry," members.aol.com/bbyars1/contents.html
- "The MacTutor History of Mathematics Archive," www-history.mcs.st-and.ac.uk/history/
- "Past Notable Women of Computing and Mathematics," www.cs.yale.edu/~tap/past-women-math.html
- "Biographies of Women Mathematicians," www.agnesscott.edu/lriddle/women
- "Great Mathematicians," physics.hallym.ac.kr/reference/physicist/mathematician.html
- "Great Minds in Mathematics," www.mcs.csuhayward.edu/~malek/Mathlinks/Minds/Minds1.html
- "Hall of Great Mathematicians," www.siue.edu/~dcollin/mathfame.html
- "Mathematicians of the African Diaspora," www.math.buffalo.edu/mad/mad0.html
- "Mathematics Contest Page," www.olemiss.edu/mathed/problem.htm
- "The Math Forum Student Center," forum.swarthmore.edu/students

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Twentieth-Century Mathematics: A Brief Review of the Century

by Lawrence Shirley

Discussing the history of mathematics in classrooms is often recommended as a way to help show that mathematics was not handed down unchanged from God (or Euclid!) into the students' notebooks but has been changing and growing throughout the centuries. We hope that students will come to see mathematics as dynamic and constantly new, not old and settled or necessarily to be accepted as it is. Most history books and sidebars in textbooks concentrate on the significant developments in mathematics made by the Egyptians, Babylonians, Greeks, and Europeans of the sixteenth to nineteenth centuries. These textbooks often avoid the developments of the twentieth century, thereby giving students the impression that mathematical progress stopped about a hundred years ago.

One reason for this omission is that it is not easy to look at the recent past with historical eyes; much of the "history" of the twentieth century is ongoing. History books have to stop somewhere, and authors often hesitate to come too close to the date of publication. Hence, most current textbooks on the history of mathematics cover little of the second half of the twentieth century. Now that the end of the century is in sight, it is appropriate for historians to look back and for teachers to show that mathematics is not only alive and well but in its most productive period ever. Historians have noted that more than half of all known mathematics has been developed since 1900; and some would even say, since 1950 (see fig. 6.1 and fig. 6.2).

Another two-pronged reason for avoiding recent history is that most school mathematics is based on the older material and much of twentieth-century mathematics is too abstract and too difficult for easy use in middle schools. Even if the actual content may be difficult, however, the exciting stories of the people, developments, results, and applications deserve coverage. These stories can add life to school mathematics and encourage our students to join in the fun! The following is a brief summary that hints at the flavor of recent mathematical developments. Apologies are offered in advance for the many omissions and oversimplifications. This century has seen too much for just a short article! Some references at the end offer guidance on finding more details.

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Applied Mathematics

Probably the most obvious achievements of mathematics in the twentieth century are in its applications. Even if students have not yet formally studied these topics, they have seen the results. Modern physics has almost become a branch of mathematics; Einstein, Bohr, Dirac, Feynman, Gell-Mann, and many other notable physicists did much of their work in mathematics. Relativity uses some of the seemingly strange theoretical abstract geometries from the previous century and shows them to be more tangible than their originators dreamed. Quantum mechanics applies probability and group theory to the structure of subatomic particles. Physics also flows into astronomy and cosmology. Stephen Hawking, a physicist and mathematician who holds the same professorial chair at Cambridge that was once held by Newton, is working to develop a "grand unified theory" of the universe. Hawking is especially interesting as an example of struggle and perseverance because his body has become wasted with Lou Gehrig's disease even as his mind has explored the universe. If physics is applied mathematics, engineering is applied physics. The many twentieth-century marvels of engineering highlight the power of mathematics in automobiles and interstate

<table>
<tr><th></th><th>Applied</th><th>Computers</th><th>Pure</th><th>Education and Society</th></tr>
<tr><td>1900s</td><td>Airplane design
Relativity</td><td>Babbage? (Oops! Wrong century!)</td><td>Hilbert's twenty-three unsolved problems</td><td>"First" mathematics educators</td></tr>
<tr><td>1910s</td><td>Electric and telephone wiring systems</td><td></td><td>Principia Mathematica</td><td></td></tr>
<tr><td>1920s</td><td>Quantum mechanics</td><td></td><td>Foundations and logic
Ring theory
Functional analysis</td><td>NCTM founded</td></tr>
<tr><td>1930s</td><td>Game theory
Atomic physics</td><td>Computability theory
Turing machine
Circuit design</td><td>Gödel's incompleteness theorem
Bourbaki
Topology</td><td>"Meaningful" mathematics</td></tr>
<tr><td>1940s</td><td>Cryptology
Manhattan Project
Simplex method</td><td>ENIAC
Programming</td><td></td><td>How to Solve It</td></tr>
<tr><td>1950s</td><td>Mathematical economics
Interstate highway system</td><td>UNIVAC
FORTRAN language</td><td>Prime numbers</td><td>Concern over weak curriculum</td></tr>
<tr><td>1960s</td><td>Aerospace engineering</td><td>COBOL, BASIC languages</td><td>Solution to continuum hypothesis</td><td>"New math" curriculum revision</td></tr>
<tr><td rowspan="2">1970s</td><td colspan="2">Computer power for statistics</td><td></td><td rowspan="2">Lakatos's philosophy</td></tr>
<tr><td>Catastrophe theory</td><td colspan="2">Four-color theorem proved with computer assistance</td></tr>
<tr><td rowspan="2">1980s</td><td colspan="2">Dynamical systems
Fractals</td><td colspan="2"></td></tr>
<tr><td></td><td>More widespread use of computers in mathematics</td><td>Packing spheres theorem</td><td>Ethnomathematics</td></tr>
<tr><td rowspan="2">1990s</td><td rowspan="2">Cosmology—string theory</td><td colspan="2">Computer networks help find more perfect numbers</td><td>NCTM's Standards</td></tr>
<tr><td></td><td>Proof of Fermat's last theorem</td><td></td></tr>
</table>

Fig.6.1. A timeline of mathematics in the twentieth century

highways; airplanes and spacecraft; telephones and television; and, of course, computers.

Two other important areas of twentieth-century applied mathematics are statistics and probability. Both had limited use before 1900, mostly in calculation, but mathematicians in both areas have built solid theoretical foundations and found many new applications. In the early part of the century, such statisticians as Pearson, Fisher, and Kendall developed analyses and methodologies, some of which now carry their names; but the tediousness of the calculations limited their applications. Computers greatly opened up the field, with not only much higher speeds of calculation but also powerful new types of analysis, including Monte Carlo methods and simulations. Incidentally, on the "low tech" side of statistics, middle school students may be interested to note that the stem-and-leaf and box-and-whisker display methods are both products of the latter part of this century.

At one time, probability seemed to be little more than a handy tool for gamblers. In the twentieth century, however, it has been applied in quantum mechanics to

Kenneth Appel	1932–	four-color theorem
Stefan Banach	1892–1945	functional analysis, topological vector spaces
Edward Begle	1914–1985	school curriculum, "new math"
"Nicolas Bourbaki"		abstract algebra, topology, deductive rigor
Paul Cohen	1934–	foundations, continuum hypothesis
Ubiratan D'Ambrosio	1933–	ethnomathematics, sociopolitical issues
George Dantzig	1914–	linear programming, simplex method
Jean Dieudonné	1906–1992	topology, abstract algebra, part of Bourbaki group
Albert Einstein	1879–1955	physics, relativity
Paul Erdös	1913–1996	number theory
Kurt Gödel	1906–1978	foundations, incompleteness of mathematics
Wolfgang Haken	1928–	four-color theorem
G. H. Hardy	1877–1947	number theory, other pure mathematics
Stephen Hawking	1942–	cosmology
David Hilbert	1862–1943	geometry, foundations, relativity
Grace Hopper	1906–1992	computers, COBOL
Imre Lakatos	1922–1974	philosophy of mathematics
Henri Lebesgue	1875–1941	theory of measure, generalized integral
Benoit Mandelbrot	1924–	fractals
John Nash	1928–	economics, equilibrium
Emmy Noether	1882–1935	relativity, abstract algebra, ring theory
George Pólya	1887–1985	problem solving
Srinivasa Ramanujan	1887–1920	number theory, function theory
Bertrand Russell	1872–1970	philosophy, foundations, logic
Atle Selberg	1917–	number theory
Reinhard Selten	1930–	economics, equilibrium
Claude Shannon	1916–	computer circuit theory
David E. Smith	1860–1944	mathematics education
René Thom	1923	catastrophe theory
Alan Turing	1912–1954	computer theory, computability
John von Neumann	1903–1957	game theory, computer programs
André Weil	1906–1998	algebraic geometry, part of Bourbaki group
Andrew Wiles	1953–	Fermat's last theorem
Grace Chisholm Young	1868–1944	set theory
Christopher Zeeman	1925–	application of catastrophe theory, dynamical systems

Fig.6.2. Some important mathematicians of the twentieth century.

the location of electrons, as noted previously, and, through John von Neumann's game theory, to strategic analyses in business, economics, politics, and warfare. John Nash and Reinhard Selten, both mathematicians, even shared the Nobel Prize for economics in 1994 for their work in this area.

In fact, mathematics has contributed to many areas of business, far beyond the Dickensian bookkeepers of the nineteenth century. Some consider George Dantzig's 1947 development of the simplex method of linear programming, a powerful tool for optimization in business, to be one of the more important mathematical achievements of the century. Around 1970, catastrophe theory, developed by René Thom and Christopher Zeeman, considered abrupt, noncontinuous results from continuous actions. The theory has applications in finance and business, as well as biology and other fields that were not previously closely related to mathematics.

The complexities of economics, biology, and other "messy" fields, such as meteorology and ecology, are now being tackled by the mathematics of chaos theory and dynamical systems. Students may have heard of chaos theory in the film Jurassic Park. The most famous example is the "butterfly effect" of Edward Lorentz, in which he suggests that the flapping wings of a butterfly in Brazil might trigger a chain of events to cause a tornado in Texas!

Computers in Mathematics

Some would urge that computer history is not part of mathematics history, but computers have closely interacted with mathematics throughout this century. One is tempted to start the story with Charles Babbage, often considered the father of computers. Even though he had twentieth-century ideas, however, Babbage's lifespan (1702–1887) was a hundred years earlier! The world had to wait until the 1930s for significant work in computers. Alan Turing's theory of computability and Claude Shannon's ideas on circuit switching gave theoretical backing to the development of programming. John von Neumann's ideas put programming input into the same format as data input, thus making computer design much more flexible. The 1950s and 1960s saw the first computer languages, such as FORTRAN, BASIC, and COBOL—the latter developed by Grace Hopper, who also coined the term *computer bug* when a moth caused her computer to break down. These languages helped bring computer use to ordinary scientists, engineers, and businesspeople. Meanwhile, mathematics continued to influence computer science theory and technology in such areas as coding theory from algebra and network analysis from topology.

Mathematicians were hesitant to embrace computers, arguing that mathematics is an endeavor of the mind, not the mechanized calculation of computers. In 1976, however, Wolfgang Haken and Kenneth Appel announced a proof of the famous four-color theorem, unsolved for over a hundred years. Although the proof is difficult, middle school students can understand the problem: on any map colored so that two neighboring countries do not have the same color along a boundary line, four colors are sufficient. A key to the solution of Haken and Appel was a detailed analysis of nearly two thousand configuration cases, performed by computer. Even using computers, the checking took over six months; without computers, it may have taken decades. Philosophically, the announcement reopened the old question of what constitutes a proof.

Computers also played a major role in the development of fractals. The concept of fractals involves geometric objects that are made of infinite patterns of increasingly smaller replications of themselves. In practice, fractals were difficult to comprehend without the ability of computer graphics to zoom deeper and deeper into their structures. The most famous is the Mandelbrot plot, which has been called the most complex mathematical object over envisioned.

Pure Mathematics

Although applications and computer work may be most visible to the public, mathematical purists would argue that abstract mathematics has been the most important growth area of this century. The century began with a "homework assignment" for mathematicians when David Hilbert, speaking at the 1900 meeting of the International Congress of Mathematicians, listed twenty-three unsolved problems that needed attention. Some still remain unsolved! Many of Hilbert's challenges dealt with the foundations of mathematics, especially an attempt to set all of mathematics in the logical framework of a system of axioms. That project was tabled in 1931, however, when Kurt Gödel proved that any axiomatic mathematical system was "incomplete"—that is, true statements could exist that could not be proved or disproved. In fact, using a complex system of numbering logical systems and statements, Gödel produced a true statement that essentially said, "This statement cannot be proved." Compare this sentence with the statement "I am lying." The idea of self-reference was later to reappear in fractals and recursion theory. Gödel's ideas also contributed to Turing's theory of computability.

Another important event was the solution of the continuum hypothesis. Georg Cantor had shown the cardinality—roughly, the number of members—of the set of rationals and the set of integers to be the same, countably infinite, labeled $\aleph_0$. He also showed the cardinality of the real numbers to be a higher uncountable infinity, which he called the continuum *c*. The question was, Does any cardinality exist between $\aleph_0$ and *c*? The continuum hypothesis said that the answer was "no." The eventual proof was a surprise. Paul Cohen showed that the answer was "it doesn't matter"; answering either "no" or "yes" could lead to a number theory without contradictions. This situation is comparable to the acceptance or nonacceptance of Euclid's fifth postulate in Euclidean versus non-Euclidean geometry, so the structures were called Cantorian and non-Cantorian number theories.

The century has also seen much work in functional analysis, abstract algebra, and topology. Functional analysis came from earlier work in linear algebra, dealing with special problems of differentiation and integration. Emmy Noether and Grace Chisholm Young were two prominent women from early in the century who worked in analysis and algebra. Although their work is esoteric for students, they are significant in showing the greater role that women have begun to play in mathematics. A stranger life story is that of Nicolas Bourbaki, whose name appears on several books that help organize and tie together many important areas of abstract algebra and topology. Bourbaki, however, never existed! The name was a collective pseudonym of a group of mostly French mathematicians who worked on "Bourbaki's" books, taking turns with the painstaking writing and editing and constantly arguing about how the ideas should be presented for greater rigor and clarity. As a joke, the mathematicians also made up a biography and genealogy of Bourbaki and even celebrated his daughter's wedding!

Topology emerged from the nineteenth century as a major new area of mathematics, linking ideas of algebra, analysis, and geometry. Informal topology appeals to students, including the older topics of Möbius rings and Königsberg bridges, but this century has seen the solution of the four-color theorem, mentioned previously; studies of knots and the space around knots; and studies involving turning spheres inside out. In the 1990s, topologists debated proofs of the optimal way of packing spheres in a closed space, although some observers noted that fruit sellers have been packing oranges in boxes for a long time!

Number theory is the oldest area of pure mathematics, dating from before Pythagoras, but it continues to flourish as a source of tantalizing questions and difficult answers. Prime numbers remain puzzling, still without any recognized pattern in their sequence. In the 1800s, Chebyshev proved that between every counting number and its double is at least one prime number. In 1930, seventeen-year-old Paul Erdös found a much simpler proof. Later, it was shown that for $n > 47$, a prime number always occurs between n and $9n/8$. In 1949, Erdös and Atle Selberg also found a more elegant proof of the distribution of prime numbers. Erdös lived until 1996 as the grand old weird man of mathematics, thinking of nothing but mathematics. He had no home but traveled from university to university with all his belongings in a small bag, helping number theorists solve more problems while they helped him wash his clothes and reminded him to stop and eat. Middle school students could investigate the Goldbach conjecture, which is still unproved, that any even number larger than 2 can be expressed as the sum of two prime numbers.

The Greeks were also intrigued by perfect numbers, which are numbers for which the sum of their proper factors equals themselves, but they found only the first four (6, 28, 496, and 8128). For centuries, new perfect numbers were discovered in only a tiny trickle because of the difficulties in calculation as much larger numbers were checked. As of 1900, only six more had been found. Mechanical calculators helped in the discovery of two more, in 1911 and 1914, respectively. Then, after 1950, computers began to help. Starting from the 1950s, six more perfect numbers were found, then five, four, and four more, respectively, in the 1960s, 1970s, and 1980s. By 1990, the new perfect numbers were in the hundreds of thousands of digits and the process seemed that it might slow again, although three more were found, in 1993, 1994, and 1996. Help came in through cooperation on the Internet, however, with hundreds of people using a new checking program. Another perfect number was found later in 1996, two more in 1997 and 1998, and probably more between this writing and its publication.

The most celebrated mathematical event of the late twentieth century was the solution of Fermat's last theorem, which had been challenging mathematicians since the 1600s. Again, even though the proof is extremely difficult, students can understand the statement: Although many whole-number solutions exist to the Pythagorean equation of $x^2 + y^2 = z^2$ (such as 3, 4, 5 and 5, 12, 13), none exist when the exponents are higher than 2. After seven years of secret work in an attic home office, Andrew Wiles announced the result in 1993, then spent two years tidying up details. His work was significant not only because it solved a longstanding problem but also because his method helped link several areas of mathematics, such as elliptic functions, modular forms, and non-Euclidean geometry, that seem to have little to do with number theory.

Mathematics Education, History, and Culture

Although students have been learning mathematics for hundreds of years, the formal study of mathematics education as an academic field did not really get started until the first decade of the twentieth century. David Eugene Smith, a mathematician working at Columbia University, recognized the need to study the learning process and curricular problems of mathematics and is often considered the first mathematics educator. Of course, many mathematics teachers were already at work, and they had begun to form professional organizations. In 1920, several groups came together to form the National Council of Teachers of Mathematics

(NCTM), which was to become the primary organization of mathematics education and the largest mathematics-related organization in the world.

Throughout the century, mathematics educators have mainly been concerned with strengthening curricular content and improving instructional practices. A central theme of both curriculum and instruction has been problem solving. George Pólya offered basic ideas about better problem-solving methods in his book *How to Solve It* (1945), which became one of the all-time best sellers of mathematics. The concerns for deeper understanding of mathematical ideas also led to the major curriculum revisions of the late 1950s and 1960s that came to be called "new math." Although "new math" represented many good ideas, many thought that it overemphasized rigor and abstraction, and its implementation was bogged down in confusing claims of various programs and competing ideas of what was best. The reaction against "new math" slowed progress, until reports of low achievement in the 1980s led to new thinking and, in 1989, to *Curriculum and Evaluation Standards for School Mathematics* from NCTM. Followed by two companion volumes and, at the end of the decade, an updated version, *Principles and Standards for School Mathematics*, the *Standards* document has provided guidance for much of the mathematics education for the new generation of the twenty-first century.

Like education, culture has always interacted with mathematics, but the last third of the twentieth century has seen new interest in this interaction. This interest is expressed in concerns over gender issues in mathematics education and in the fields of mathematics and science in general and is reflected in the new field of ethnomathematics, which looks specifically at mathematics in various cultures. Equity in mathematics education seeks to ensure that the field of mathematics benefits from the talents of all and that the power of mathematics is available to all. A review of this article alone reveals that although issues of equity are now being addressed, opportunities in mathematics have largely been limited to white males, even in the twentieth century. We hope that mathematics in the twenty-first century can draw from the wider population.

Increased interest in the history and philosophy of mathematics has helped put mathematical developments into a broader context and has encouraged thought about where mathematics may be going next. One philosopher in particular, Imre Lakatos, advocated a more flexible view of the meaning of mathematics. He suggested that proofs and mathematical concepts are not final and static but constantly being reevaluated and undergoing dynamic change. In fact, this review was written for that reason—to remind us all that mathematics did not stop in the nineteenth century but is ready to make the jump into the twenty-first!

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Some excellent resources have been collected at the following sites on the World Wide Web. Take some time to explore their many useful links.

The Sci.Math FAQ Team. *Frequently Asked Questions in Mathematics* (trivia and historical information). www.cs.unb.ca/~alopez-o/math-faq/math-faq.html

Trinity College, Dublin. *The History of Mathematics* (collection of files and links). www.maths.tcd.ie/pub/HistMath/Hist Math.html

University of Saint Andrews, Scotland. *MacTutor History of Mathematics Archive.* www-history.mcs.stand.ac.uk/history/

Woltman, George. *Great Internet Mersenne Prime Search* (primes and perfect numbers). www.mersenne.org

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A few videotapes are also available that relate to this article.

Csicsery, George Paul. *N Is a Number: A Portrait of Paul Erdös* [a visit with the late number theorist], 1993.

Lynch, John, directed by Simon Singh. *The Proof* [background to Andrew Wiles's proof of Fermat's last theorem]. Episode from the *Nova* series. BBC-TV/WGBH-Boston coproduction, 1997.

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Generative Topics: Building a Curriculum around Big Ideas

by Ron Ritchhart

DATA from the Third International Mathematics and Science Study (TIMSS) point out "a comparative lack of focus and coherence in the American mathematics curriculum" and an absence of "meaningful connections between the big ideas of mathematics" (Schmidt 1997). These findings, along with those reported in other international comparisons (e.g., Stevenson and Stigler [1992]), suggest that we must carefully examine the actual content of both our written and our implemented mathematics curriculum to be sure that it gives students the focus and coherence they need to develop mathematical understanding. Although teaching methodology is important, no issue is as fundamental to the heart of teaching or as important to reform in mathematics education as our decision about what to teach.

What we teach is more than a list of topics—addition, subtraction, measurement, place value, and so on. What we teach encompasses the flavor, focus, and spin that we place on these topics. For example, teaching place value as a scheme for organizing and recording quantities is fundamentally different "content" from teaching place value as a set of ordered names to be matched appropriately to a series of columns. Consequently, two teachers working from the same syllabus can teach radically different content depending on the emphasis, context, and applications that they give each topic. This subtle difference between "topics" and actual "content" makes *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) challenging to implement. Merely teaching the topics mentioned in the *Standards* document does not necessarily translate into achieving the vision of the Standards or providing a coherent curriculum.

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For systemic reform efforts to be effective, this question of what to teach must be answered in a meaningful way by many people—administrators, curriculum specialists, parents, policy makers, and teachers. However, individual teachers charged with implementing the curriculum have an immediate need and opportunity to wrestle with this question. Although curriculum specialists and research groups offer teachers guidance and well-developed resources, effective teaching requires more than transmitting the plans, ideas, or knowledge of others. Even the best-laid-out curriculum or a well-planned unit may fail if teachers do not understand the essence of what they are teaching, bring passion to the topic, and make the content meaningful. Teachers play an important decision-making role in the actual implementation of curriculum, making the connections that can turn a fragmented curriculum of skills, facts, and knowledge into a coherent quest for understanding.

So how can teachers focus their curriculum and identify powerful mathematical content?

This article explores one productive avenue by which teachers, as well as principals and supervisors, may grapple both collectively and individually with the question of what to teach. Through examples of teachers' exploring the generative nature of traditional curriculum topics, this article shows how teachers can come to a better understanding of why an individual topic is important, how that topic helps build mathematical power, and what opportunities exist for making connections and supporting transfer. A framework for discussing curriculum and reframing a traditional mathematics unit is given, as well as a discussion of how this refocusing can change instruction.

Where to Start?

The NCTM's *Standards* document is an excellent starting place, focusing on big ideas at the core of mathematics: problem solving, communication, reasoning, and connections. These "habits of the mathematician" suggest both a methodology for our teaching and a framework for students' learning of mathematics, and act as the glue that holds the curriculum together. Although the Standards offer a loose curricular framework by specifying important broad mathematical strands and diverse topics, individual teachers must personalize and develop their own sense of curricular ownership to be effective. We must seek clarity for ourselves before we can impart it to our students.

Imagine that you have just been hired to teach second grade at a new school. At the opening faculty meeting, your principal informs you that the money for textbooks has been set aside to buy resources and materials for teaching. No standardized tests are planned. Instead, the school will decide how best to assess students' understanding and convey this information to parents. Later that afternoon, you sit down with your new colleagues to plan the mathematics curriculum. How do you decide what is important to teach?

Such a scenario may either excite or terrify you. Although it may be removed from your present situation, every year more and more schools engage in just such an enterprise on one level or another. Unfortunately, in the face of such enormous opportunity, many teachers either look back to what they had done previously or look to what others are doing, rather than look forward to what can give students real power.

The following conversation took place among a group of urban teachers from Denver, Colorado, involved in redesigning a mathematics unit on measurement for their combination first-and second-grade classrooms. The group begins the search for a generative topic by discussing what is involved in the process of measuring. The teachers reflect on how measurement is used and what it looks like in its simplest forms. For the moment, they leave behind their notions of what they had done in past measurement units, their favorite lessons, and even the district's objectives. They will address those elements later. Their first goal is to open up their perceptions of the topic.

Carl: Well, first you have to decide what to measure, don't you?

Joan: No, I don't think so. You have to have a reason for measuring first. You want to find something out, and you have to use measurement. It has a lot to do with the context, with the situation.

Lee: It seems like when you measure, you're often comparing something, like to see if it fits or if you have enough in comparison to what you need. That provides both your reason and your decision about what to measure.

Carl: There are lots of things that could be measured. That's important to think about. You would need to be aware of these different things. You could measure length, area, volume, weight, ...

Juanita: This goes back to having a reason for measuring and then deciding which of these things or characteristics to measure. Area, length, and volume are like attributes you choose from.

Lee: But a lot of the time you don't even have to measure, I don't, and I don't think kids do, either. If you want to find out if someone's taller or if something fits, you just compare them. There's often no reason to go any further than comparing.

Joan: You're right, but that comparing is still measuring. You're finding out something about the two objects. If you're comparing to see who is taller, then you're dealing with height.

Carl: But is it measurement if you don't have a number to go with it?

Juanita: Sure it is. I can tell that my coffee mug holds more than your mug by measuring visually. I think you're measuring whenever you're making a comparison.

Lee: Maybe that's our generative topic, comparing. It seems to capture what we do and what kids do naturally when you measure. It's a lot more informal and natural than how I've traditionally approached measurement. My big question now is, Where do the numbers come into all of this?

Carl: Well, there are times when you need to measure and get some kind of number to quantify that measurement. Like if you're buying wallpaper, you measure your wall and then buy the paper. If you had the paper already, you might not need to measure.

Joan: That's if you had a lot of paper; with my luck I'd run out halfway through. In your example of wallpaper, you can't directly compare things, so you have to measure and get a number.

Lee: That's the idea of transitivity that we read about. Remember? It was in that book of research ideas. If "a" is longer than "b" and "b" is longer than "c," then we know "a" is longer than "c." But we have to remember that the concept of transitivity is developmental. All of our kids might not be there yet.

Juanita: But I think we can still talk about it, since it really points up the need for measurement by comparing things you can't just match up or see automatically. I think kids can think about the idea of comparing right away. It's something that they do naturally. All we have to do is help focus their attention on the different attributes or characteristics that can be measured.

Joan: It's a new way for me to think and teach about measurement, but you're right, I think kids will find it more accessible. I also like all of the connections we've talked about already: attributes, transitivity, numbers .

Through their discussion, the teachers come to a better understanding of the processes and contexts of measurement. In the process, they find a more generative way in which to frame the unit—as a unit about making comparisons. Perrone (1994) describes a generative topic as "those ideas, themes, and issues that provide the depth and variety of perspectives that help students develop significant understandings." Perkins (1992) puts forth three essential criteria by which the generativity of a topic can be judged: centrality to the discipline, richness of connections, and accessibility to the students. A generative topic, therefore, is the origin from which investigations are spawned, new concepts and understandings are produced, and connections are created. A generative topic is the parent idea pregnant with possibilities for explorations.

The idea of a generative topic as a driving force of curriculum has a long tradition in education. Dewey (1964) and Whitehead (1929) advocated teaching essentials that were relevant to children and captured their natural curiosity. The Coalition of Essential Schools finds generativity through posing essential questions (Cushman 1989). For our group of teachers, essential questions for their unit might be "How can we compare objects?" or "What does it mean to talk about less and more?" Finally, the National Science Foundation–funded project Teaching to the Big Ideas has been involved in working with elementary teachers to understand and teach the "big ideas" of mathematics (Edwards 1994).

Why a Generative Topic, Not a Theme?

Teachers have long taught mathematics and other subjects through themes. Although a theme can make a topic more accessible and interesting to students, it does not meet the criteria of a generative topic because themes lack centrality to the discipline. For example, a teacher may decide to use the theme of whales. The unit might involve measuring whales, locating whales on a map with a coordinate grid, collecting data on whales, and so on. A lot of mathematics is present here, but not a central mathematical idea. Themes are vehicles for applying skills, and perhaps even learning new skills, but they pull away from the discipline of mathematics itself. Mathematics is used as a tool to learn about something else. Such work can be a valuable and an engaging learning experience for students as long as teachers keep a tight rein on the theme and do not let it overwhelm the disciplinary content.

In comparison, a generative topic is concerned first and foremost with the core ideas of the discipline, building disciplinary understanding by focusing on these central ideas. Figure 7.1 presents a series of questions to help a teacher better understand the generative nature of any standard curricular topic, such as probability, estimation, fractions, and so on. A generative topic should capture the essence and spirit of what the subject or topic is all about, why it is important, and how it is used. Identifying the central core of a mathematics topic requires looking beyond the basic skill level of traditional mathematics instruction to explore how it is used in real life. Through a generative form of the topic, students and teachers should feel invited to actively engage in and apply the topic. It should make the topic accessible by connecting to students' knowledge of the world. By being accessible, a generative topic is also motivating. Students can connect the topic to their experience and engage in its application immediately. A generative topic—whether expressed as a

Reframing a Traditional Unit

1. Centrality: Why is this topic important?
    - What does it allow us to do?
    - Where can it take us?
    - Humankind invented _________ to solve what problem or to facilitate what activities?
2. Accessibility: What is this topic really all about?
    - How would you describe this topic in everyday language as opposed to mathematics language?
    - How would students invent or discover this topic on their own? In what context?
    - What do you do when you engage ______________?
    - How does this topic appear in real life?
3. Richness: How is this topic connected to other topics in and outside of mathematics?
    - What skills and knowledge must you draw on as you apply and use this topic?
    - In what other areas is this topic used?
    - Understanding of this topic opens up what new areas?

Fig.7.1. Questions to consider in creating a generative topic

question, a word, or a phrase—should encourage understanding, extensions, and connections.

Teaching with a Generative Topic

On the surface, the creation of a generative topic appears to be only a cosmetic change. After all, what does it matter what we call it if it is still the same unit? The power of generative topics lies in their ability to help teachers reconceptualize what they are teaching and view it from a fresh perspective. The generative topic helps identify both the real-life connections and the connections with other mathematical topics. Teachers must then act on these connections if the use of generative topics is to make a difference in students' formation of connections and transference of knowledge and skills.

By focusing students' attention on making comparisons from the start, the teachers in our example move beyond an emphasis on the acquisition of discrete measurement skills to a richer, more dynamic conception of measurement. Rather than teach students measurement techniques for determining length or volume, the teachers want to lead their students to recognize that an object may have many attributes that can be measured or compared. The teachers know that they will need to help their students acquire the language to talk about and identify these attributes.

To help students focus on measurement attributes and develop measurement vocabulary, the teachers began the unit with a series of whole-class lessons that had students explore how two objects are the same and how they are different, for example, a pencil and a comb. As students listed the many attributes, they recorded them on the chalkboard to help build vocabulary. During this initial activity, length was mentioned as an attribute, but not width. This activity was repeated with several other objects to develop students' observation skills.

The next day, the comparison question was modified. Instead of looking at "same and different," students were asked, "What does this object have more of than does this one?" Students were inventive in their answers as they compared the comb and the pencil—more color, more teeth, more points. Finally, someone said, "It is more tall." The teacher seized this opportunity to discuss the observation with the students: "Yes, this one is taller. We can say that the comb has more height than the pencil. What about looking across? If up and down is *height*, we sometimes call across *width*. Does the comb have more width than the pencil?" As students explored the more-less game with various objects, they gradually became adept at noticing attributes having to do with measurement—height, width, weight, area, and volume. Students came to see that most objects can be measured on multiple fronts.

The notion of comparing also helped the teachers remain focused on the meaningful application of measurement as a means of comparison. The teachers planned to put students in circumstances in which they must determine rather than be told what attribute to measure. To move beyond a direct comparison of two objects, the teachers wanted to design situations in which objects could not be directly compared. Students needed to develop some sort of quantitative measure. To motivate students, the teacher played the more-less game with two immovable objects, a mark on the chalkboard and a poster taped to the wall. Initially, students estimated to say that one had more length than the other, but disputes arose in the room. The teacher challenged the students to determine the answer definitively. As if by magic, students spontaneously began to measure by using their forearms, pencils, a sheet of paper, or a train of Unifix cubes. The dispute was resolved, and the realistic context of measurement was established.

Throughout the unit, students engaged in what Perkins (1992) calls "understanding performances," in which they actively applied, constructed, generalized, and contextualized their understanding as opposed to practicing rote skills. Since the teachers view learning as a product of thinking, they reviewed activities and lessons, asking themselves, "Will students need to think to do this lesson, or will they merely be repeating and mimicking my actions without understanding?" Some favorite and seemingly successful activities were rejected or modified to stimulate more thinking. For example, the teachers decided that it was not appropriate to ask children to measure the length of their desks without any reason, so they abandoned the worksheet that guided students through such an activity.

In watching students work, the teachers clearly saw that they were very comfortable with the idea of direct comparison and were becoming more familiar with the new language of measurement. The teacher posed a new question for students to explore: "You said that this object has more length. How much more?" A series of rich discussions ensued in which students moved from relying on informal notions to devising a plan for comparing the objects by using beans, pennies, and Unifix cubes. Teachers modified this question to encourage students to explore a variety of measurement attributes.

Assessing a Generative Topic

In the past, the measurement units that the teachers in Denver had taught did not have a strong assessment component. The work usually progressed from one measuring-and-recording activity to another, with the culmination being yet one more similar activity. Because the teachers were not clear about what they wanted their students to understand about measurement and did not know how their activities built toward such an understanding, both ongoing and summative assessment suffered. The focus was on measuring accurately rather than on understanding the concept and the multifaceted nature of measurement. Although the previous assessments told what students could do—measure correctly—they did not tell whether students recognized when to measure and what to measure.

Using their new understanding of measurement to direct their efforts, the teachers decided that at the end of the unit, students should be able to pick any two objects and tell multiple ways in which those two objects might be compared. Some of those attributes should be attributes of measurement. Students should then be able to identify "how much more" one object had on any number of measurement-related attributes. To prepare students for this activity, the teachers introduced new measurement activities involving measuring area with tiles and measuring volume with cubes.

In their final assessment, students selected two objects and compared them in as many ways as possible. They used their data to determine which was bigger—a task that they now understood requires a judgment. Finally, they justified their selection with a written response. Having been actively immersed in the process of making such comparisons and judgments, the students were able to participate in an assessment that mirrored the instruction and fostered a high probability of success. The element of choice gave students an opportunity to show their imagination and to exercise ownership over the problem. In evaluating the work, the teachers made their own estimates and looked closely at students' recording. It was evident who had a rich conception of measurement and who was still comfortable exploring only length and width.

Making Multiplication and Division More Generative

A group of third- and fourth-grade teachers discussed the topic of multiplication. Quickly they came to the consensus that multiplication is really a means of counting things in groups. This notion opened up the conversation, and they were soon talking about division. In thinking about grouping situations, the teachers realized that many problems can be solved by either division or multiplication, depending on how a person thinks about the situation.

They noted that their district curriculum had divided these topics but that they could be reunited through the generative topic of "grouping." In thinking about multiplication and division in this way, they recognized that the concepts are accessible to all their students. Grouping relates to students' lives in many ways and offers rich opportunities for exploration. By moving away from their traditional skills-oriented approach and toward a problem-solving emphasis, the teachers were more comfortable teaching the unit to their multiage classes. As planning progressed, the teachers sought ways to put students in situations that would require them to group objects—such as sharing a box of crackers among friends or counting out a large quantity.

Initial experiences involved the class members in finding different ways to line up in rows of equal length. As students explored this problem with manipulatives, the teachers observed them dealing with the concepts of factors, primes, remainders, addition, and subtraction. From the start, students made rich and meaningful connections while developing confidence in problem solving. Many such experiences were explored, and soon the term and the symbol for multiplication were introduced as shorthand to be used instead of writing out "rows of," "groups of," or "stacks of." With a better understanding of multiplication and division, the teachers were able to devise a unit that involved students in challenging problems from the beginning—a dramatic departure from the way that they had previously taught multiplication.

Conclusions

Generative topics are one avenue for opening up the curriculum, creating connections, and providing means for transfer. They also help refocus and reorient the mathematics curriculum in a way that is consistent with current calls for systemic reform (Ewing and Sharp 1996; Stevenson and Stigler 1992). In addition, generative topics respond to Whitehead's (1929) call of seventy years ago to "let the main ideas which are introduced into a child's education be few and important." However, it is important to remember that much of the power of generative topics lies in the dialogue surrounding their formation. By discussing the mathematics they teach and seeking to discover why it is important, how it is used, and its basic essence, teachers can move beyond teaching discrete and often isolated skills. Simply renaming a unit is not enough to change its focus. The generative topic should invite a more open-ended exploration of the topic that emphasizes new goals in terms of understanding and application.

A move toward the use of more generative topics in mathematics helps us realize the vision of the *Standards* document (1989) by placing greater emphasis on mathematics as connections, problem solving, reasoning, and communication. The connections are evident in the topic itself and are an integral part of instruction. Students are invited to learn through problem solving as they explore and rediscover the topic in a natural context. As this exploration expands and students engage in understanding performances, a sense of inquiry is developed in which both questioning and reasoning about the topic are encouraged and enhanced. Throughout, students are encouraged to build on their everyday language as they gradually integrate the appropriate mathematics vocabulary into their discussions.

Through the regular use of generative topics, students develop a sense of curiosity about, and appreciation of, mathematics while acquiring skills within the context of their application rather than in isolation. Through generative topics, both students and teachers can explore and exploit a world of deep, rich, and powerful mathematics that has both focus and coherence.

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ACTIVITIES

The Equity and Curriculum Principles

Zaslavsky: "Ethnomathematics and Multicultural Mathematics Education"
Morrow: "Ensuring That All Children are Powerful Technology Users"
Moldavan: "Observing the Power of Matt, the Mathematician"

Write a reaction paper to these three short articles. Discuss the following.

- How do the articles relate to particular aspects of the Equity Principle?
- Describe any of the authors' points with which you agree or disagree strongly.
- Discuss any difficulties that you think might exist in carrying out a classroom implementation of at least one of the recommendations made by the authors.

Rock and Shaw: "Exploring Children's Thinking about Mathematicians and Their Work"

Group Project

Pose the following three questions, given in the article, to elementary, middle school, high school, and/or college-age students. Compile the responses, and comment on your findings.

1. What do mathematicians do?
2. What types of problems do mathematicians solve?
3. What tools do mathematicians use?

Shirley: "Twentieth-Century Mathematics: A Brief Review of the Century"

Discuss how the information contained in the article did or did not change your perception of what mathematics is and what mathematicians do.

2.

THE TEACHING, LEARNING, AND ASSESSMENT PRINCIPLES

EVEN though the eight articles in this section have each been listed under a particular heading, any one of them could serve as an illustration for one of the other principles. The three principles provide a set of complimentary lenses for examining the complex and interactive nature of the classroom teaching-learning experience.

The Teaching Principle

EFFECTIVE mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.

Effective teaching requires—

- different kinds of mathematical knowledge—knowledge about the whole domain, and flexible knowledge about general curriculum goals and grade-level specifics;
- knowledge of how students learn mathematics, how particular ideas can be presented to students, and how to assess students' understanding;
- the establishment and nurturing of an environment conducive to learning mathematics;
- continuing efforts to learn about mathematics and pedagogy, and to reflect on and refine instructional practice through interactions with students, colleagues, and professional development. (NCTM 2000, pp. 16–19)

In the overview of the *Principles and Standards* document, the teacher is characterized as being—

> responsible for creating an intellectual environment in the classroom where serious engagement in mathematical thinking is the norm. Effective teaching requires deciding what aspects of a task to highlight, how to organize and orchestrate the work of students, what questions to ask students having varied levels of expertise, and how to support students without taking over the process of thinking for them. (NCTM 2000b, p. 4)

The articles in this group illustrate the complexity of this role. In the first article, Smith examines the teacher's role in supporting and structuring students' work during problem-solving activities. Through a classroom vignette, the author illustrates how difficult it can be for a teacher to not intervene too quickly when students encounter frustration with unfamiliar tasks. This particular teacher, through reflection on her teaching activities, gained valuable insights into how her expectations of student behavior needed to change as she implemented new instructional materials.

In the second article, Kazemi contrasts the way in which two different teachers orchestrate classroom discourse related to the same problem-solving task. A difference was found in the kinds of questions that the teachers asked when soliciting students' explanations of their solution strategies. One teacher continually "pressed" the students to provide mathematical reasons as part of their explanations, whereas the other teacher required only that the students describe the procedural steps that they had used to reach their solutions. The author states that differences in classroom expectations can affect students' achievement in both problem solving and conceptual understanding.

In the final article, Whitenack and others use a short vignette of a primary-grade student's incorrect addition strategy to examine the effect of social influences on students' mathematical understanding. The analysis of the vignette suggests that the particular student may have constructed her understanding of addition procedures by noting which strategies were highly valued by her teacher rather than by connecting strategies with particular mathematical structures. The authors encourage teachers to consider social explanations as well as cognitive reasons when accounting for their students' mathematical activity.

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Redefining Success in Mathematics Teaching and Learning

by Margaret Schwan Smith

A major goal of current reform efforts is to help students learn mathematics with understanding. "Good" mathematical tasks are an important starting point for developing mathematical understanding, but selecting and setting up good tasks does not guarantee a high level of student engagement (Smith and Stein 1998). Using such tasks can, and often does, present challenges for teachers and students.

Teachers, accustomed to establishing rules and procedures for students to follow, may have difficulty with a more nebulous set of responsibilities aimed at supporting students as they construct mathematical knowledge. Students may become frustrated with tasks that they cannot immediately solve and pressure the teacher to show them "how to do it." As a result, "good" tasks are often carried out in ways that remove the opportunities for problem solving and sense making (Doyle 1988; Stein, Grover, and Henningsen 1996) and reduce students' chances to engage in meaningful learning of mathematics (Stein and Lane 1996).

This situation occurs because teachers think that frustration and lack of immediate success are indicators that they have somehow failed their students. Teachers have no way to measure their own success when teaching is no longer defined as providing explanations and procedures or evaluating students' learning by the correctness of their solutions. According to J. P. Smith (1996), more traditional "teaching and telling" gives teachers a sense of efficacy—a perception that they have had a positive impact on students' learning—that is undermined by current efforts to reform mathematics instruction. Smith suggests that we need to establish "new moorings" for efficacy that are closely related to reform-oriented teaching to ensure that teachers will sustain their commitment to new ways of teaching.

What would constitute these "new moorings" to which Smith refers, and how will teachers come to establish these new indicators of success? Elaine Henderson (name changed), a teacher who participated in the QUASAR project, is an interesting case for exploring this question. QUASAR, which stands for Quantitative Understanding: Amplifying Student Achievement and Reasoning, was a national project funded by the Ford Foundation to improve mathematics instruction for students attending middle schools in economically disadvantaged communities (Silver and Stein 1996; Silver, Smith, and Nelson 1995). A study of Henderson's teaching practice during her first year of implementing this innovative mathematics curriculum revealed that supporting students' engagement with "good" mathematical tasks required her to redefine what it meant for both herself and her students to be successful in mathematics class (Smith 2000). Over time, Henderson began to establish for herself new indicators that she was, in fact, having a positive impact on students' learning. I encourage you to reflect on your own experiences as you consider Henderson's practice at the beginning and end of her first year of "doing reform" in light of her changing views of success.

Before Reform

Before her involvement in the QUASAR project, Henderson's goal in teaching mathematics

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This manuscript is based on the author's dissertation. An upcoming article in the Elementary School Journal *offers a more extended discussion of Elaine Henderson and her learning during her first year of enacting an innovative curriculum (Smith 2000). The author wishes to acknowledge the helpful comments of Marjorie Henningsen on a previous draft of this article.*

was to ensure that her students would successfully learn the algorithms that they needed. She explained the procedures to be learned, demonstrated a small number of sample problems, monitored students' completion of a few problems, and had students work individually on a larger set of similar problems, using the strategies that she had taught.

Henderson had always considered herself to be a successful mathematics teacher—her students did well on the district standardized tests, teachers in subsequent grades who had her students always remarked that they were well prepared, and parents often requested that their children be placed in her classroom. Henderson saw herself as someone who related well to students and was able to motivate them to learn. She was always on the lookout for new ways to help her students experience success.

Henderson made the decision to participate in the QUASAR project in spring 1990 because she was intrigued by its approach to teaching mathematics, which emphasized thinking and reasoning and encouraged collaboration among students. She believed that such an approach could be beneficial to students, and she was eager to try something new. Henderson had always liked problem solving and found the tasks in the new curriculum to be more interesting than ones that she had typically used in the past. She also liked the fact that the project would give her the opportunity to work closely with her colleagues in mathematics and with teacher educators at a local university who would support the teachers' implementation efforts.

Making a Change

Henderson spent several weeks working through the new curriculum with her school and university colleagues and found that she learned much about mathematics in the process. She was eager to give her students the same opportunity for learning. Early in the school year, however, Henderson began to notice that students were struggling with the tasks in the new curriculum. If they could not solve a problem immediately, which was often the case, they would just say, "I don't know" and give up. Henderson was concerned that her students were not actively participating in class. She believed that if students were not involved, they would not learn and that if they did not feel successful, they would lose their motivation to stay involved. As Henderson commented, "[I will] guarantee them success by asking them to do things they couldn't fail to do right. I can't ignore that success breeds success. Too many are starting out with what I'm sure they perceive to be failure."

To ensure success, Henderson began altering problems from the curriculum. At times, she put in an extra step or took out something that she thought was too hard; she rewrote problem instructions to be clearer and created easier problems to lead up to more challenging ones. In addition, during classroom instruction, Henderson often "broke a task down" into a set of subtasks, each of which the students could complete successfully. Henderson believed that the more correct answers a student gets, the greater the learning he or she experiences. She tended to ask questions that were, as she explained, "not designed for deep thinking, just success."

A lesson on exploring patterns that took place in late October furnishes an example of how Henderson started supporting students' involvement with the new tasks. The curriculum materials suggested providing students with the first three trains in a pattern sequence, then asking students to build the fourth train. Next, the students were asked to build a larger train in the sequence, such as the tenth or fifteenth, without building all the trains in between, then discuss why they believed that the tenth or fifteenth train looked as it did. She presented the pattern shown in figure 8.1 and gave student partners eleven minutes to write down what they noticed about each of the trains, which was a deviation from what the curriculum materials suggested. Henderson then asked students to share their observations orally and at times sent a student to the overhead projector to demonstrate an observation using overhead pattern tiles.

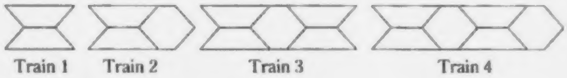

Fig.8.1. Train pattern

Some of the observations made by students were nonmathematical (e.g., "[the first train] looks like a squished pop can," "[the third train] looks like a belt buckle"). Others showed a basic understanding of a mathematical concept that pertained to only one train (e.g., "[the first train] has four equal sides," "[the second train has] two trapezoids and a hexagon"). A few attempted to look across trains (e.g., "they will have two [sides] that are equal and four [sides] that are equal," "trapezoids are double the number [of the train, if you replace the hexagons with trapezoids]"). Through teacher questioning, two generalizations emerged during the lesson: (1) for the even-number trains, the number of trapezoids was equal to the train number and (2) for the even-number trains, the number of hexagons was half the train number. Consider the following excerpt from the class in which Henderson tries to make sure that all her students "see" this pattern:

Ms. H: Let's do a couple more. Listen to what he's saying, and see if you can do it also. Charles, in train 12, how many red trapezoids will there be?
Charles: Twelve.
Ms. H: And then how many hexagons will there be?
Charles: Six.
Ms. H: Can you do it for any even number I give you? Tony, what's he doing?
Tony: On the hexagons, he's doing the same number as the problem.
Ms. H: On the what? Trapezoids or hexagons?
Tony: Trapezoids.
Ms. H: The number of trapezoids is the same number as the trains.
William: And the hexagons, he's taking it in half.
Ms. H: Take it in half for the hexagons, and it's the same for the other. OK, let's everybody try one. I can pick any even number: train 50. How many trapezoids will there be? Everybody!
SS: Fifty.
Ms. H: How many hexagons?
SS: Twenty-five.
Ms. H: Train 20, how many trapezoids?
SS: Twenty.
Ms. H: How many hexagons?
SS: Ten.
Ms. H: Train 100, how many trapezoids?
SS: One hundred.
Ms. H: How many hexagons?
SS: Fifty.
Ms. H: All right, look up there. On train 50, will it end with two trapezoids or will it end with a hexagon?
SS: Hexagon.
Ms. H: How do you know that? What up there tells you that? Michele, I haven't heard from you in a while.
Michele: The odds are the trapezoid, and evens are the hexagon.
Ms. H: So you're saying to me that any number I tell you, you can tell me how it ends? Eleven.
Michele: It's odd. It would be trapezoid.

When the class was over, Henderson commented, "The lesson was all I could have asked of the kids . . . it was very exciting." She thought that the students had remained engaged throughout the period, the choral responses gave all the students an opportunity to feel good about themselves, and everyone seemed to be able to predict future trains using the generalizations.

Reflecting on the Lesson a Few Weeks Later

At regular intervals over the course of the year, the classes of Henderson and her colleagues were videotaped. The videotapes and related observations were part of the ongoing efforts of the QUASAR project to document classroom practices. The videotapes were also used by teachers to reflect on their practices. At a staff development session, Henderson volunteered to show the clip of the videotaped segment from the October lesson featuring the pattern-block sequence shown in figure 1 and share their thoughts about the lesson. She commented that she had discovered that her students were not very good at observations, but she thought that they were definitely verbalizing more when she "broke it down" for them, focusing their attention on each part of a sequence rather than on the entire series.

One of the university teacher educators who was working with Henderson and her colleagues asked Henderson whether the students had progressed to the point at which they could make observations without her "breaking it down." The educator asked Henderson how long teachers need to break tasks down for students and whether Henderson thought that some of the observations would come out naturally if the students were given time and opportunity to develop them. Henderson responded that she hoped she did not always have to structure things but that her students were still at the "comfort stage" and needed this support. Once students experienced success, Henderson believed that they would try harder and would no longer need to have their learning experiences structured in this way.

The exchange with the educator appeared to have a powerful impact on Henderson. In her journal later that day, she reflected, "I need to make sure I'm not structuring too much. It is easy to be too leading and feel OK about it because the kids seem happy. After all, many kids are happy with 'shut up and add.' " Over time, Henderson continued to question her approach, wondering whether structuring the learning opportunities to ensure success would help students become competent and confident problem solvers—a central goal of the new curriculum—or whether the structuring would remove important challenging aspects from the students' mathematical experiences.

In an attempt to provide more problem-solving opportunities, during a quiz a few days later, Henderson asked students to divide a square into four congruent parts in five different ways. After students had found two solutions, they claimed that the problem could not be solved in other ways and started to give up. Henderson encouraged them to try different things to find the answer and explained to them "that they weren't really doing problem solving until they reached the point where they thought they couldn't do it." As she noted, "I'm used to the idea that their confusion means that I haven't introduced the lesson properly or have given kids something too hard. Sometimes that is true, but sometimes it is necessary to go through panic before we find solutions." Later that month, Henderson watched the entire videotape of the October pattern train lesson again and wrote the following entry in her journal:

> Students had ample opportunity to successfully predict visual pattern-block trains in this lesson, but it was set up too much for success and not enough for the frustration that goes with problem solving. Unfortunately, the lesson contained too much whole-group teacher questioning and students' explaining and not enough time for students to stretch and discover independently/collaboratively. I made the lesson safe for the kids—no fail—which was my goal at the time. I now think I need to let them go through the frustration that goes with problem solving. The lesson probably wouldn't have looked as smooth, but I think it would have stretched the kids more. I am at a different point in my thinking than I was at the time of the lessons.

Redefining Success

Throughout the year, Henderson continued to struggle with her feelings of failure, concerns about her students' need for success, and her commitment to giving her students genuine problem-solving opportunities. For Hendérson, redefining success required that she create new expectations for students of what it means to know and do mathematics and new expectations for her role in supporting students' learning. She came to believe that daily success should be measured by the extent to which students meet the expectations that she set for them and the extent to which she supported students in meeting those expectations. Henderson was effectively establishing "new moorings" for her own teaching of mathematics.

A series of lessons that Henderson conducted the next spring offers some insight into how these new moorings played out in her classroom. When the students were working in small groups or pairs on challenging problems, Henderson encouraged them to use diagrams and sketches as tools for solving problems, insisted that they be able to explain how they figured out problems, and encouraged them to consider alternative strategies. Many of Henderson's questions focused on trying to understand what students were doing as they solved the problems (e.g., "How did you get that?" "Why do you think that?" "What's happening here?") and on encouraging students to communicate (e.g., "Can you explain it to him?" "Why don't you ask your partner what she was doing first?"). When students presented their solutions at the overhead projector, the class was responsible for understanding the solutions and asking questions that would illuminate any errors in the approach. The teacher's periodic statement, "I assume that everyone understands if you have no questions," was a way of reminding students that the responsibility for understanding was theirs. Students assumed that they had given correct answers and had appropriately justified their solutions if no questions arose from their peers or the teacher, and this assumption was, in fact, true.

Although some problems were particularly difficult for students, rather than guide them step-by-step through the frustration, Henderson would ask one student to begin a problem at the overhead projector but not complete it. Henderson monitored the progress of the class but left her students to make sense of the presenter's explanation and determine how it would further their work on the problem.

Table 1 summarizes Henderson's new expectations for her students, her approach to supporting students in meeting the new expectations, and what she took as classroom-based evidence that students were successful. Classroom-based indicators of success, such as those found in table 1, present early evidence that

TABLE 8.1
Key Elements in Henderson's Efforts to Redefine Success for Herself and Her Students

NEW EXPECTATIONS FOR STUDENTS	TEACHER ACTIONS CONSISTENT WITH EXPECTATIONS	CLASSROOM-BASED INDICATORS OF SUCCESS
Most "real" tasks take time to solve; frustration may occur; perseverance in the face of initial difficulty is important.	Use "good" tasks; explicitly encourage students to persevere; find ways to support students without removing all the challenges in a task.	Students engaged in the tasks and did not give up too easily. The teacher supported students when they "got stuck" but did so in a way that kept the task at a high level.
Correct solutions are important, but so is being able to explain how you thought about and solved the task.	Ask students to explain how they solved a task. Make sure that the quality of the explanations is valued equally as part of the final solution.	Students were able to explain how they solved a task.
Students have a responsibility and an obligation to make sense of mathematics by asking questions when they do not understand and by being able to explain and justify their solutions and solution paths when they do understand.	Give students the responsibility for asking questions when they do not understand, and have students determine the validity and appropriateness of strategies and solutions.	With encouragement, students questioned their peers and provided mathematical justifications for their reasoning.
Diagrams, sketches, and hands-on materials are important tools for students to use in making sense of tasks.	Give students access to tools that will support their thinking processes.	Students were able to use tools to solve tasks that they could not solve without them.
Communicating with others about your thinking during a task makes it possible for others to help you make progress on the task.	Ask students to explain their thinking, and ask questions that are based on students' reasoning, as opposed to how the teacher is thinking about the task.	Students explained their thinking about a task to their peers and the teacher. The teacher asked probing questions based on the student's thinking.

changes in practice have occurred, which is a necessary first step before changes in student learning outcomes can be expected. In Henderson's situation, data collected on student performance showed that her students had grown over the year, as measured by a standardized test and an innovative performance assessment. By the end of the school year, Henderson thought that students were meeting the expectations that she had set for them and that she had done a good job in building a scaffolding for her students to support high-level thinking and reasoning.

Sharing Your Reflections

The purpose of this article is to raise questions about what it means to be successful as a teacher and as a student in reform-inspired mathematics classrooms. For Henderson, this questioning meant a yearlong process of reflection on her practice and a redefinition of the factors that contribute to success. It is important to note that Henderson was not alone on her journey. She had the invaluable support of her teacher and university colleagues. I encourage you to discuss with your colleagues the issues raised in this article and to share through the "Teacher to Teacher" department in *Mathematics Teaching in the Middle School* your perspective on what it means to be successful.

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Discourse That Promotes Conceptual Understanding

by Elham Kazemi

As MATHEMATICS teachers, we want students to understand mathematics, not just to recite facts and execute computational procedures. We also know that allowing students to explore and have fun with mathematics may not necessarily stimulate deep thinking and promote greater conceptual understanding. Tasks that are aligned with the NCTM's curriculum standards (NCTM 1989) and that are connected to students' lives still may not challenge students to build more sophisticated understandings of mathematics. The actions of the teacher play a crucial role.

This article presents highlights from a study that demonstrates what it means to "press" students to think conceptually about mathematics (Kazemi and Stipek 1997), that is, to require reasoning that justifies procedures rather than statements of the procedures themselves. This study assessed the extent to which twenty-three upper elementary teachers supported learning and understanding during whole-class and small-group discussions. "Press for learning" was measured by the degree to which teachers (1) emphasized students' effort, (2) focused on learning and understanding, (3) supported students' autonomy, and (4) emphasized reasoning more than producing correct answers. Quantitative analyses indicated that the higher the press in the classroom, the more the students learned.

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Like researchers in other studies (e.g., Fennema et al. [1996]), we observed that when teachers helped students build on their thinking, student achievement in problem solving and conceptual understanding increased. To understand what press for learning looks like in classrooms, we studied in depth two classes with higher scores for press and two classes with lower scores, and we looked closely at mathematical activity and discourse in the classes. The high-press classroom of Ms. Carter is contrasted with the low-press classroom of Ms. Andrew.

Students in Ms. Carter's and Ms. Andrew's classes were exploring the concept of equivalence and the addition of fractions. They worked on fair-share problems, such as the following:

> I invited 8 people to a party (including me), and I had 12 brownies. How much did each person get if everyone got a fair share? Later my mother got home with 9 more brownies. We can always eat more brownies, so we shared these out equally too. This time how much brownie did each person get? How much brownie did each person eat altogether? (Corwin, Russell, and Tierney 1990, p. 76)

Similarities between Classrooms: Social Norms

In both Ms. Carter's and Ms. Andrew's classes, we saw students huddled in groups, materials scattered about them, figuring out how to share a batch of brownies equally among a group of people. The students seemed to be engaged in and enjoying their work. Often each group found a slightly different strategy to solve the problem. After moving from group to group, listening to and joining student conversations, both teachers stopped group activity to ask students to share their work and explain how they solved the problem.

The NCTM *Standards* document supports the view that social norms—practices such as explaining thinking, sharing strategies, and collaborating that we see in both classrooms—afford opportunities for students to engage in conceptual thinking. Many teachers establish those social norms in their classrooms quite readily. But social norms alone may not advance students' conceptual thinking.

Differences between Classrooms: Sociomathematical Norms

Although Ms. Andrew and Ms. Carter both valued problem solving and established the same social norms in their classrooms, important differences were seen in the quality of their students' engagement with the mathematics. To understand those differences, we looked more closely at the norms that guide the quality of mathematical discourse, the *sociomathematical* norms (Yackel and Cobb 1996). Teachers and students actively negotiate the sociomathematical norms that develop in any classroom. Sociomathematical norms identify what kind of talk is valued in the classroom, what counts as a mathematical explanation, and what counts as a mathematically different strategy. In the brownie problem, for example, students grapple with ideas of equivalence, part-whole relations, and the addition of fractional parts. Sociomathematical norms help us understand the ways in which fraction concepts are supported within the context of sharing and explaining strategies.

Through our study of the four classrooms, we identified four sociomathematical norms that guided students' mathematical activity and helped create a high press for conceptual thinking:

- Explanations consisted of mathematical arguments, not simply procedural summaries of the steps taken to solve the problem.
- Errors offered opportunities to reconceptualize a problem and explore contradictions and alternative strategies.
- Mathematical thinking involved understanding relations among multiple strategies.
- Collaborative work involved individual accountability and reaching consensus through mathematical argumentation.

Other norms may also contribute to a high press, but these norms captured the major differences in the way that mathematics was treated by the high- and low-press teachers.

Explaining strategies

The following examples illustrate some of the differences in the two classrooms. First, in Ms. Carter's class, explanations were not limited to descriptions of steps taken to solve a problem. They were always linked to mathematical reasons. In the following example, Ms. Carter asked Sarah and Jasmine to describe their actions and to *explain why* they chose particular partitioning strategies.

Sarah: The first four we cut them in half. [Jasmine divides squares in half on an overhead transparency. See fig. 9.1..]

Ms. Carter: Now as you explain, could you explain why you did it in half?

Sarah: Because when you put it in half, it becomes four . . . four . . . eight halves.

Ms. Carter: Eight halves. What does that mean if there are eight halves?

Sarah: Then each person gets a half.

Ms. Carter: Okay, that each person gets a half. [Jasmine labels halves 1 through 8 for each of the eight people.]

Sarah: Then there were five boxes [brownies] left. We put them in eighths.

Ms. Carter: Okay, so they divided them into eighths. Could you tell us why you chose eighths?

Sarah: It's easiest. Because then everyone will get . . . each person will get a half and [addresses Jasmine] "How many eighths?"

Jasmine: [Quietly] Five-eighths.

Ms. Carter: I didn't know why you did it in eighths. That's the reason. I just wanted to know why you chose eighths.

Jasmine: We did eighths because then if we did eighths, each person would get each eighth, I mean one-eighth out of each brownie.

Ms. Carter: Okay, one-eighth out of each brownie. Can you just, you don't have to number, but just show us what you mean by that? I heard the words, but . . . .

[Jasmine shades in one-eighth of each of the five brownies that were divided into eighths.]

Jasmine: Person one would get this . . . [points to one-eighth].

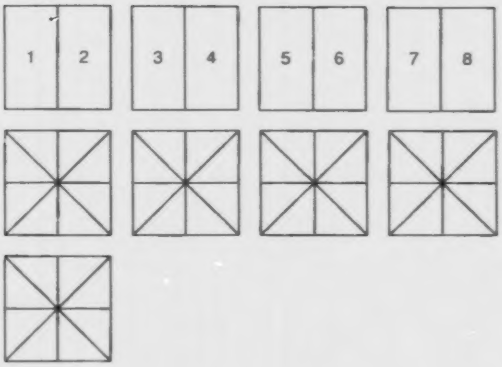

Fig.9.1. Sharing nine brownies among eight people

Ms. Carter: Oh, out of each brownie.

Sarah: Out of each brownie, one person will get one-eighth.

Ms. Carter: One-eighth. Okay. So how much did they get if they got their fair share?

Jasmine and Sarah: They got a half and five-eighths.

Ms. Carter: Do you want to write that down at the top, so I can see what you did?

[Jasmine writes 1/2 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 at the top of the overhead transparency.]

The exchange among Sarah, Jasmine, and Ms. Carter highlighted the conceptual focus of the lesson on fair share. Ms. Carter asked Sarah to explain the importance of having eight halves and why the partitioning strategy using eighths made sense. After Jasmine gave a verbal justification, Ms. Carter continued to press her to link her verbal response to the appropriate pictorial representation—by shading the pieces—and to the symbolic representation—by writing the sum of the fractions.

The same degree of press did not exist in Ms. Andrew's classroom. Ms. Andrew's students engaged in the same social practice of sharing their strategies with the class, but the mathematical content of classroom conversations was different. Students shared solutions by giving procedural summaries of the steps they took to solve the problem, as demonstrated by the following exchange, in which Raymond described his solution for sharing twelve brownies among eight people. Ms. Andrew had drawn twelve squares on the chalkboard.

[Raymond divides four of the brownies in half.]

Ms. Andrew: Okay, now would you like to explain to us what . . . loud . . . .

Raymond: Each one gets one, and I give them a half.

Ms. Andrew: So each person got how much?

Raymond: One and one-half.

Ms. Andrew: One-half?

Raymond: No, one and one-half.

Ms. Andrew: So you're saying that each one gets one and one-half. Does that make sense? [After a chorus of "yeahs" comes from students, Ms. Andrew moves on to another problem.]

Unlike Ms. Carter, Ms. Andrew did not ask her students to justify why they chose a particular partitioning strategy. Instead, Ms. Andrew often asked questions that required a show of hands or yes-no responses, such as "How many people agree?" "Does this make sense?" or "Do you think that was a good answer?" Ms. Andrew wanted to engage her students in the activity and to see if they understood, but the questions she asked yielded general responses without revealing specific information about the students' thinking.

Reacting to mathematical errors

By emphasizing mathematical reasons for actions, Ms. Carter created opportunities for her students to prove that their solutions were correct. She resisted telling students that an answer or reason was wrong, and she invited others to respond to incorrect solutions. Ms. Carter modeled the kinds of questions that may help students think through their own confusion by using their existing knowledge. Those questions usually involved graphical representations of the fractions. In small groups, students challenged one another when they disagreed on a solution and helped one another find errors.

The interaction among Ms. Carter, Jasmine, and Sarah continued with the following conversation.

[Jasmine writes 1/2 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 at the top of the overhead transparency.]

Ms. Carter: Okay, so that's what you did. So how much was that in all?

Jasmine: It equals 1 1/8 or 6/8.

Ms. Carter: So she says it can equal 6 and 6/8? [She misheard Jasmine.]

Jasmine: No, it can equal 6/8 or it can equal 1 1/8.

Ms. Carter: Okay, so you have two different answers. Could you write them down so people can see it? And boys and girls, I'd like you to respond to what they've written up here. She says it either could equal 6/8 or 1 1/8.

Ms. Carter: Matthew had his hand up and was thinking about it. Someone from team 5. Anybody from team 6 that has a response? Right now. I'm just going to let you look a minute. See if anyone has a response. Andrew, you had your hand up, is that right? [waits] Still only have four hands up. I wonder if you're all looking up here and seeing? She's given us two answers here, 6/8 or it can equal 1 1/8. Okay, could those four people right now . . . do you agree with both answers?

Students: No . . . .

Ms. Carter: Do you have a reason why you don't agree? Don't explain it to me, but do you have a reason? Raise your hand if you have a reason why you don't agree. [hands] One, two, three, four, five, six . . . okay. Would those six people please stand? Okay. Would you please, you're going to be in charge of explaining why you don't agree to your team. [She assigns those six students to teams.] Right now, if you don't agree would you please tell them what you think the answer is and why you don't agree. Go ahead. Explain.

Ms. Carter could have stepped in and pointed out why 6/8 and 1 1/8 are not equal. Instead, her response to this mistake was to encourage her students to explore the error by providing the conceptual reasons for why 6/8 and 1 1/8 are not equal. She engaged the entire class in thinking about which solution was correct instead of talking with only the two presenters or correcting their mistake herself, and she created an opportunity for her students to practice articulating their thinking.

The mistake also created an opportunity for the entire class to explore contradictions in the solution and to build an understanding of fractional equivalence and the addition of fractions by using an area model. This type of activity and discourse was typical in Ms. Carter's classroom. In a whole-class discussion, each group shared its proof that 1 1/8 was correct. Neither the students nor Ms. Carter belittled, penalized, or discredited anyone who made a mistake. The atmosphere of mutual respect between the students and Ms. Carter allowed the class to think about and build conceptual understandings eagerly.

Ms. Andrew treated errors differently. Note how she provides the mathematical reasoning when three boys explained their solution for sharing five brownies among six people.

Ms. Andrew: They got 1/2, you already said that. And then 1/6 and then another sixth. So, how many sixths did they get?

Anthony: One, two.

Ryan: One, two.

Joe: 1/12.

Ms. Andrew: What did you say? [to Joe] They got two . . .

Ryan: Sixths.

Anthony: 2/12.

Joe: 2/6.

Ms. Andrew: 2/6 [confirming the right answer]. Why did you say 2/12? Because there are twelve parts altogether?

Anthony: Yeah.

Ms. Andrew: Okay, be sure not to get confused. Because there are two brownies not one. Perfect. Good, good job.

At first, the boys appeared to be guessing the answer to Ms. Andrew's question. She focused on Joe once he stated the right answer. Although she predicted accurately why Anthony said 2/12, she did not ask him to think about why his answer did not work. Instead, she asked and answered the question herself and did not press Anthony to sort out his confusion. Her statement "Because there are two brownies, not one" was left unexplained. As this example illustrates, limited opportunity was available for the members of the group to engage in conceptual thinking about what 1/6 and 1/12 signify and how the graphical representation is linked to the numeric representation.

Both Ms. Carter and Ms. Andrew allowed students to make mistakes. That social norm, however, was not enough to press students to examine their work conceptually. Both teachers wanted their students to learn from their mistakes, but Ms. Andrew often supplied the conceptual thinking for her students. In Ms. Carter's class, inadequate solutions served as entry points for further mathematical discussion.

Comparing strategies

Students in both classrooms worked together, shared their strategies, and were praised for their efforts. Students in both classrooms attended to nonmathematical similarities between shared solutions, such as the layout of the paper or the uses of color. In Ms. Andrew's class, strategies were typically offered one after the other, with discussion limited to nonmathematical aspects of students' work. For example, a pair of students noted that they cut the paper brownies and pasted the pieces under stick-figure illustrations.

Another pair had drawn lines from the fractional parts of the brownies to the individuals that received them. Although the partitioning strategy in both was the same, students viewed the strategies as different because the representations were different. Ms. Carter, however, pressed her students to go beyond their initial observations and reflect on the mathematical similarities and differences between strategies.

Accountability and consensus

In inquiry-based classrooms, students often work together to share interpretations and solutions and construct new understandings. Important differences arose between Ms. Andrew's and Ms. Carter's classes in the way in which they emphasized individual accountability and consensus. Ms. Carter required her students to make sure that each person contributed to, and understood the mathematics involved in, the group's solution. If students disagreed about an answer, she encouraged them to prove their answers mathematically and to work until they arrived at a consensus. If she noticed that students were not listening to others during an activity, she reminded them that they had to prove their solutions and that each group member must be prepared to discuss the reasons for the solution in front of the class. As a result, the distribution of work was more equitable. Students listened to one another's ideas and evaluated their appropriateness before using them.

Ms. Andrew did not describe and discuss collaboration beyond the general directive to "work with a partner" or "remember to work together." Neither individual accountability nor consensus emerged as topics of discussion in whole-class activity. Typically, only one person would be in control of group work at any particular time and would complete most of the work.

Conclusion

We saw a consistently higher press for conceptual thinking in Ms. Carter's class. She took her students' ideas seriously as they engaged in building mathematical concepts. In both whole-class discussions and small-group work, all students were accountable for participating in an intellectual climate characterized by argument and justification. Four sociomathematical norms governed mathematical discourse in Ms. Carter's classroom: explanations were supported by mathematical reasons, mistakes created opportunities to engage further with mathematical ideas, students drew mathematical connections between strategies, and each student was accountable for the work of the group.

When teachers create a high press for conceptual thinking, mathematics drives not only the activities but the students' explanations as well. As a result, student achievement in problem solving and conceptual understanding increases.

Action Research Ideas

- Over time, listen for differences in the number of times that you that interrupt a student's explanation, restate a student's explanation, or provide a solution strategy. By keeping a daily log, notice any changes in the nature and quantity of your responses.
- *(a)* Identify the social norms and the sociomathematical norms that characterize your classroom. *(b)* Discuss the issue of sociomathematical norms with a colleague. Share your goals and the problems that you expect to encounter. Continue to discuss your progress with your colleague over time. Encourage your colleague to engage in a similar program to create a higher press. *(c)* Observe and discuss each other's teaching.
- *(a)* Reflect on the discourse associated with a problem recently discussed in your classroom. Using a four-point scale from 0 (low press) to 4 (high press), rate the discourse according to each of the sociomathematical norms that characterize Ms. Carter's classroom. *(b)* Set personal goals for each of the sociomathematical norms. Use such questions as the following to help establish a high press: "How can you prove that your answer is right? Can you prove it in more than one way? How is your strategy mathematically different from, or mathematically like, that of [another student]? Do you agree or disagree with [another student's] solution? Why? Why does [strategy *x*] work? Why does [strategy *y*] not work?" *(c)* After four weeks, reevaluate your classroom, using the same scale and the same sociomathematical norms. Note your areas of improvement, and set new goals for the next four weeks.

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Using Technology to Foster Teachers' Reflections about Children's Arithmetical Thinking

by Joy W. Whitenack, Nancy Knipping, Sue Novinger, Linda Coutts, and Barbara Reys

TWENTY-EIGHT primary-grade teachers, working in pairs, stared intently at the screens of their computers as they watched a CD-ROM containing QuickTime movie excerpts of students responding to mathematical tasks during an interview. These teachers were participating in a year-long professional-development project on ways to promote young students' mathematical thinking. The project consisted of three phases.

During phase 1, the teachers met for four three-hour sessions to discuss issues related to problem solving and creating a classroom atmosphere conducive for all students to learn. During phase 2, the summer seminar, the teachers engaged in mathematics explorations and watched and analyzed examples of students' problem solving depicted on the CD-ROM. They considered the mathematical big ideas that students construct and explored various instructional activities that they might implement. In phase 3, the teachers met biweekly for eight three-hour sessions in the fall to continue exploring ways to foster students' mathematical development. They also engaged in small-group discussions about videotapes of interviews that they had conducted and excerpts of their teaching.

One assignment during the summer seminar involved working with a partner to develop a mini–case study of one or more students' responses to interview tasks, such as basic facts, paper-and-pencil computation, and problem-solving situations, on the CD-ROM. This package included excerpts of six students' interviews and allowed teachers to access segments in several ways. For example, a pair of teachers might explore several students' solutions for a particular type of task, say, addition of single-digit number sentences. Alternately, they might choose to examine excerpts of one student's solution methods across various tasks. We wanted this work to help teachers reflect on students' arithmetical thinking and to develop hypotheses about their understanding.

As a culminating experience, the teachers shared their findings on the final day of the seminar. The following discussion focuses on issues that the teachers

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addressed in sharing their case studies. We highlight how the CD-ROM's interview segments afforded the teachers opportunities to explore students' thinking and to consider ways in which the learning environment, including their teaching actions, might facilitate the students' mathematical learning.

Examining Children's Thinking

In their mini–case presentations, the teachers offered many conjectures about the students' different solution methods. Some conjectures were descriptions of the students' thinking. For instance, the teachers spoke of whether a student had solved a task by counting on. They also spoke of the relative sophistication of the students' methods and interpretations. For example, one pair of teachers suggested that two students, Nate and Tyrone, seemed to have different understandings of number concepts. They noted that Nate described his solution to 5 – 2 by putting up five fingers and explaining, as he put down one finger and then another finger, "I'm taking away the five, I'm taking away the four. So . . . the three is left, the two is left, and the one is left." They contrasted Nate's counting activity with Tyrone's use of his fingers to solve a similar subtraction task. For 10 – 3, Tyrone first put up all ten fingers, put down three fingers on his second hand, looked at the remaining fingers, and immediately said, "Seven." The teachers explained that Tyrone, unlike Nate, could solve tasks without counting by ones. They suggested that although both students used their fingers, Tyrone thought of numbers as quantities, whereas Nate seemed to think of each finger as a particular number word, that is, the thumb is called "one," the first finger is called "two," and so on.

As this pair of teachers shared their mini–case study, they mentioned that their thinking had changed as they continued to revisit the interview segments. Not until they had viewed the segment several times did they begin to see subtle but important distinctions in the two students' understanding of number. As one teacher stated, "[Nate and Tyrone] were at very different developmental levels. . . . Tyrone is at the next step. . . . Nate is going to come to that . . . [as] he moves on."

Examining Issues Relating to the Classroom Learning Environment

During their presentations, several teachers also considered issues related to the classroom learning environment, particularly as they thought about the explanations by six-year-old Annie. The following excerpts occurred during the first several minutes of her interview. After Annie had correctly solved 3 + 3, the interviewer asked about 3 + 4. Annie said the answer was five, and then explained her thinking:

Interviewer: Okay, five. What were you thinking about that time?
Annie: I know four plus . . . two plus two is four. Add one more on, and that's five.
Interviewer: Add one more on and that would be five.

The interviewer then asked Annie if she could solve the horizontal number sentence 4 + 2:

Annie: [Three-second pause during which Annie silently mouths "five, six."] Six.
Interviewer: Oh, so what were you thinking about that time?
Annie: I knew three plus three was six. Um [six-second pause], I knew two plus two was four…
Interviewer: Um-hmm.
Annie: Add [two-second pause] two more on, and that's six.
Interviewer: Ah, you didn't count when you solved that problem?
Annie: [Shakes head no.]

Later, the interviewer placed five plastic cows on the table in front of a toy barn and indicated that more cows were in the barn. The interviewer asked Annie to determine how many cows were in the barn. The following discussion ensued:

Interviewer: Ah, so there are five out there. Well, do you know what?
Annie: What?
Interviewer: I have eight all together. I'm the farmer, and I really have eight cows. Where do you think the other cows are?
Annie: In the barn?
Interviewer: I think they are in the barn. Can you figure out how many cows are in the barn?
Annie: [Thirty-nine-second pause.] Nine.
Interviewer: Okay, talk to me about that one.
Annie: I knew four plus four was eight. Add one more on, and that's nine.

Project teachers were intrigued by Annie because she reminded them of children in their own classrooms. They were curious about the logic behind Annie's answers. They asked themselves why Annie continued to use doubles, whether she had number sense, and how a classroom teacher might help Annie develop more reasonable strategies. Annie's interview segments became a very important part of the teachers' discussions.

Annie's responses caused the teachers to see the need to go beyond explaining her methods in terms of her conceptual understanding. Instead, they explained them in terms of her understanding of what it meant to know and do mathematics. That is, they spoke of Annie's obligations as a student in her mathematics classroom.

More specifically, the teachers were impressed by the possible social influences that may have given rise to Annie's consistent use of doubles strategies. For example, one of the teachers, Judy, suggested that Annie may have used doubles because these methods were particularly valued by the classroom teacher and the fellow students. Several other teachers noted that they had thought along similar lines and concurred with Judy's explanation. Referring to the task 4 + 2, in which Annie counted on from four, as she mouthed the numbers five and six, to solve the task but used doubles, 2 + 2 and 3 + 3, to explain her answer, the teachers conjectured that Annie realized that she needed to explain her reasoning. However, she apparently did not link this obligation with her responsibility to give explanations that reflected her own thinking. Thus, Annie's solution methods made sense from a social, but not a mathematical, standpoint.

The teachers also considered reasons that their own students might use methods similar to the ones that Annie used. Annie was able to count on, yet during the interview she was reluctant to use counting strategies. This situation prompted the teachers to consider their roles and, more particularly, whether they contributed to their students' confusion by valuing only the more sophisticated methods their students used.

This issue was important for the project teachers because it illuminated the challenge that teachers face as they try to design learning opportunities for students who are functioning at various levels. Although teachers must begin with their students' current understandings, they also want to encourage more "sophisticated" thinking. Shifts in students' thinking can occur as teachers devote time to students' explanations. For many teachers, the number of students in a class means that not everyone can share thoughts every day, but decisions regarding who should do so on a particular day and how to respond to those comments, if at all, are more than management issues.

The teacher may ask a student to share his or her solution methods with others because the teacher realizes that such opportunities can help the student clarify his or her thoughts. Another student may be asked to share because the explanation can give classmates an opportunity to consider an alternative way to interpret and solve a task. For example, most first-grade teachers would welcome an explanation in which a student says that 7 + 6 = 13 "because 6 + 6 = 12, and if I add one more to one of the sixes, I need to add one more to twelve to get the answer." The teacher might highlight this explanation because the student used a strategy other than counting. In so doing, however, the teacher must be careful not to send the message that only noncounting strategies are valued. One teacher made this point when she remarked, "I'm thinking, now, of all the kids I 'made famous' for the neat strategies they used during investigations, but I never made anyone famous for counting." This comment highlighted the teachers' concerns about the importance of accepting less sophisticated methods, such as counting, when those methods are the only ones a student understands, while encouraging students to use more sophisticated strategies when they make sense.

Summary

The foregoing examples illustrate the kinds of issues that teachers raised in their discussions of the minicases. The discussion has shown how interview segments, such as Nate's and Tyrone's, gave teachers opportunities to examine the nature and the quality of students' arithmetical thinking. We have also shown how interview segments such as Annie's presented occasions for the teachers to explore notions related to their own practice and their students' experiences during teaching. Annie's interview, in particular, allowed the teachers to move beyond purely cognitive accounts to consider social explanations for students' mathematical activity.

Our summer experiences have caused us to become increasingly aware of the important role that technology might play as a catalyst for teachers to reflect on their own teaching experiences against the backdrop of students' arithmetical problem solving. Whereas using technology to support teachers' professional development was a secondary emphasis of the teacher-enhancement project, we believe that such experiences have a promising future.

The Learning Principle

STUDENTS must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.

"In the twenty-first century, all students should be expected to understand and be able to apply mathematics" (NCTM 2000, p. 20). The Learning Principle recognizes that—

- conceptual understanding is an important component of proficiency, along with factual knowledge and procedural facility;
- learning with understanding makes subsequent learning easier, facilitates dealing with novel problems and settings, and supports the development of autonomous learners;
- students have a natural desire to understand what they learn and have a considerable knowledge base on which to build;
- learning with understanding can be built by actively engaging students in tasks and experiences designed to deepen and connect their knowledge. (NCTM 2000, pp. 20–21)

Students must be active participants in the learning process. As the articles in this group illustrate, given the opportunity, students can and will engage in meaningful and significant mathematical activity. Although the first article is written from the perspective of the teacher, Reinhart's discussion of how he changed his approach from teacher-directed to student-centered instruction paints a vivid picture of the student's role in the learning process. The author reminds his students that participation is not optional. He turns the responsibility for learning back to his students by using structured groups, asking questions rather than lecturing, and eliciting multiple solutions to each problem.

The second article illustrates the power of students' invented strategies to promote meaningful learning. Warrington and Kamii describe fifth- and sixth-grade students' invented strategies for solving a series of fraction multiplication problems. The students were allowed to build on their informal knowledge of fractions and to construct methods that made personal sense as they dealt with increasingly difficult problems.

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Never Say Anything a Kid Can Say

by Steven C. Reinhart

AFTER extensive planning, I presented what should have been a masterpiece lesson. I worked several examples on the overhead projector, answered every student's question in great detail, and explained the concept so clearly that surely my students understood. The next day, however, it became obvious that the students were totally confused. In my early years of teaching, this situation happened all too often. Even though observations by my principal clearly pointed out that I was very good at explaining mathematics to my students, knew my subject matter well, and really seemed to be a dedicated and caring teacher, something was wrong. My students were capable of learning much more than they displayed.

Implementing Change over Time

The low levels of achievement of many students caused me to question how I was teaching, and my search for a better approach began. Making a commitment to change 10 percent of my teaching each year, I began to collect and use materials and ideas gathered from supplements, workshops, professional journals, and university classes. Each year, my goal was simply to teach a single topic in a better way than I had the year before.

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Before long, I noticed that the familiar teacher-centered, direct-instruction model often did not fit well with the more in-depth problems and tasks that I was using. The information that I had gathered also suggested teaching in nontraditional ways. It was not enough to teach better mathematics; I also had to teach mathematics better. Making changes in instruction proved difficult because I had to learn to teach in ways that I had never observed or experienced, challenging many of the old teaching paradigms. As I moved from traditional methods of instruction to a more student-centered, problem-based approach, many of my students enjoyed my classes more. They really seemed to like working together, discussing and sharing their ideas and solutions to the interesting, often contextual, problems that I posed. The small changes that I implemented each year began to show results. In five years, I had almost completely changed both what and how I was teaching.

The Fundamental Flaw

At some point during this metamorphosis, I concluded that a fundamental flaw existed in my teaching methods. When I was in front of the class demonstrating and explaining, I was learning a great deal, but many of my students were not! Eventually, I concluded that if my students were to ever really learn mathematics, they would have to do the explaining, and I, the listening. My definition of a good teacher has since changed from "one who explains things so well that students understand" to "one who gets students to explain things so well that they can be understood."

Getting middle school students to explain their thinking and become actively involved in classroom discussions can be a challenge. By nature, these students are self-conscious and insecure. This insecurity and the effects of negative peer pressure tend to discourage involvement. To get beyond these and other roadblocks, I have learned to ask the best possible questions and to apply strategies that require all students to participate. Adopting the goals and implementing the strategies and questioning techniques that follow have helped me develop and improve my questioning skills. At the same time, these goals and strategies help me create a classroom atmosphere in which students are actively engaged in learning mathematics and feel comfortable in sharing and discussing ideas, asking questions, and taking risks.

Questioning Strategies That Work for Me

Although good teachers plan detailed lessons that focus on the mathematical content, few take the time to plan to use specific questioning techniques on a regular basis. Improving questioning skills is difficult and takes time, practice, and planning. Strategies that work once will work again and again. Making a list of good ideas and strategies that work, revisiting the list regularly, and planning to practice selected techniques in daily lessons will make a difference.

Create a plan. The following is a list of reminders that I have accumulated from the many outstanding teachers with whom I have worked over several years. I revisit this list often. None of these ideas is new, and I can claim none, except the first one, as my own. Although implementing any single suggestion from this list may not result in major change, used together, these suggestions can help transform a classroom. Attempting to change too much too fast may result in frustration and failure. Changing a little at a time by selecting, practicing, and refining one or two strategies or skills before moving on to others can result in continual, incremental growth. Implementing one or two techniques at a time also makes it easier for students to accept and adjust to the new expectations and standards being established.

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1. Never say anything a kid can say! This one goal keeps me focused. Although I do not think that I have ever met this goal completely in any one day or even in a given class period, it has forced me to develop and improve my questioning skills. It also sends a message to students that their participation is essential. Every time I am tempted to tell students something, I try to ask a question instead.

2. Ask good questions. Good questions require more than recalling a fact or reproducing a skill. By asking good questions, I encourage students to think about, and reflect on, the mathematics they are learning. A student should be able to learn from answering my question, and I should be able to learn something about what the student knows or does not know from her or his response. Quite simply, I ask good questions to get students to think and to inform me about what they know. The best questions are open-ended, those for which more than one way to solve the problem or more than one acceptable response may be possible.

3. Use more process questions than product questions. Product questions—those that require short answers or a yes or no response or those that rely almost completely on memory—provide little information about what a student knows. To find out what a student understands, I ask process questions that require the student to reflect, analyze, and explain his or her thinking and reasoning. Process questions require students to think at much higher levels.

4. Replace lectures with sets of questions. When tempted to present information in the form of a lecture, I remind myself of this definition of a lecture: "The transfer of information from the notes of the lecturer to the notes of the student without passing through the minds of either." If I am still tempted, I ask myself the humbling question "What percent of my students will actually be listening to me?"

5. Be patient. Wait time is very important. Although some students always seem to have their hands raised immediately, most need more time to process their thoughts. If I always call on one of the first students who volunteers, I am cheating those who need more time to think about, and process a response to, my question. Even very capable students can begin to doubt their abilities, and many eventually stop thinking about my questions altogether. Increasing wait time to five seconds or longer can result in more and better responses.

Good discussions take time; at first, I was uncomfortable in taking so much time to discuss a single question or problem. The urge to simply tell my students and move on for the sake of expedience was considerable. Eventually, I began to see the value in what I now refer to as a "less is more" philosophy. I now believe that all students learn more when I pose a high-quality problem and give them the necessary time to investigate, process their thoughts, and reflect on and defend their findings.

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Share with students reasons for asking questions. Students should understand that all their statements are valuable to me, even if they are incorrect or show misconceptions. I explain that I ask them questions because I am continuously evaluating what the class knows or does not know. Their comments help me make decisions and plan the next activities.

Teach for success. If students are to value my questions and be involved in discussions, I cannot use questions to embarrass or punish. Such questions accomplish little and can make it more difficult to create an atmosphere in which students feel comfortable sharing ideas and taking risks. If a student is struggling to respond, I move on to another student quickly. As I listen to student conversations and observe their work, I also identify those who have good ideas or comments to share. Asking a shy, quiet student a question when I know that he or she has a good response is a great strategy for building confidence and self-esteem. Frequently, I alert the student ahead of time: "That's a great idea. I'd really like you to share that with the class in a few minutes."

Be nonjudgmental about a response or comment. This goal is indispensable in encouraging discourse. Imagine being in a classroom where the teacher makes this comment: "Wow! Brittni, that was a terrific, insightful response! Who's next?" Not many middle school students have the confidence to follow a response that has been praised so highly by a teacher. If a student's response reveals a misconception and the teacher replies in a negative way, the student may be discouraged from volunteering again. Instead, encourage more discussion and move on to the next comment. Often, students disagree with one another, discover their own errors, and correct their thinking. Allowing students to listen to fellow classmates is a far more positive way to deal with misconceptions than announcing to the class that an answer is incorrect. If several students remain confused, I might say, "I'm hearing that we do not agree on this issue. Your comments and ideas have given me an idea for an activity that will help you clarify your thinking." I then plan to revisit the concept with another activity as soon as possible.

Try not to repeat students' answers. If students are to listen to one another and value one another's input, I cannot repeat or try to improve on what they say. If students realize that I will repeat or clarify what another student says, they no longer have a reason to listen. I must be patient and let students clarify their own thinking and encourage them to speak to their classmates, not just to me. All students can speak louder—I have heard them in the halls! Yet I must be careful not to embarrass someone with a quiet voice. Because students know that I never accept just one response, they think nothing of my asking another student to paraphrase the soft-spoken comments of a classmate.

"Is this the right answer?" Students frequently ask this question. My usual response to this question might be that "I'm not sure. Can you explain your thinking to me?" As soon as I tell a student that the answer is correct, thinking stops. If students explain

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their thinking clearly, I ask a "What if?" question to encourage them to extend their thinking.

Participation is not optional! I remind my students of this expectation regularly. Whether working in small groups or discussing a problem with the whole class, each student is expected to contribute his or her fair share. Because reminding students of this expectation is not enough, I also regularly apply several of the following techniques:

1. Use the think-pair-share strategy. Whole-group discussions are usually improved by using this technique. When I pose a new problem; present a new project, task, or activity; or simply ask a question, all students must think and work independently first. In the past, letting students begin working together on a task always allowed a few students to sit back while others took over. Requiring students to work alone first reduces this problem by placing the responsibility for learning on each student. This independent work time may vary from a few minutes to the entire class period, depending on the task.

After students have had adequate time to work independently, they are paired with partners or join small groups. In these groups, each student is required to report his or her findings or summarize his or her solution process. When teams have had the chance to share their thoughts in small groups, we come together as a class to share our findings. I do not call for volunteers but simply ask one student to report on a significant point discussed in the group. I might say, "Tanya, will you share with the class one important discovery your group made?" or "James, please summarize for us what Adam shared with you." Students generally feel much more confident in stating ideas when the responsibility for the response is being shared with a partner or group. Using the think-pair-share strategy helps me send the message that participation is not optional.

A modified version of this strategy also works in whole-group discussions. If I do not get the responses that I expect, either in quantity or quality, I give students a chance to discuss the question in small groups. On the basis of the difficulty of the question, they may have as little as fifteen seconds or as long as several minutes to discuss the question with their partners. This strategy has helped improve discussions more than any others that I have adopted.

2. If students or groups cannot answer a question or contribute to the discussion in a positive way, they must ask a question of the class. I explain that it is all right to be confused, but students are responsible for asking questions that might help them understand.

3. Always require students to ask a question when they need help. When a student says, "I don't get it," he or she may really be saying, "Show me an easy way to do this so I don't have to think." Initially, getting students to ask a question is a big improvement over "I don't get it." Students soon realize that my standards require them to think about the problem in enough depth to ask a question.

4. Require several responses to the same question. Never accept only one response to a question. Always ask for other comments, additions, clarifications, solutions, or methods. This request is difficult for students at first because they have been conditioned to believe that only one answer is correct and that only one correct way is possible to solve a problem. I explain that for them to become better thinkers, they need to investigate the many possible ways of thinking about a problem. Even if two students use the same method to solve a problem, they rarely explain their thinking in exactly the same way. Multiple explanations help other students understand and clarify their thinking. One goal is to create a student-centered classroom in which students are responsible for the conversation. To accomplish this goal, I try not to comment after each response. I simply pause and wait for the next student to offer comments. If the pause alone does not generate further discussion, I may ask, "Next?" or "What do you think about __________'s idea?"

5. No one in a group is finished until everyone in the group can explain and defend the solution. This rule forces students to work together, communicate, and be responsible for the learning of everyone in the group. The learning of any one person is of little value unless it can be communicated to others, and those who would rather work on their own often need encouragement to develop valuable communication skills.

6. Use hand signals often. Using hand signals—thumbs up or thumbs down (a horizontal thumb means "I'm not sure")—accomplishes two things. First, by requiring all students to respond with hand signals, I ensure that all students are on task. Second, by observing the responses, I can find out how many students are having difficulty or do not understand. Watching students' faces as they think about how to respond is very revealing.

7. Never carry a pencil. If I carry a pencil with me or pick up a student's pencil, I am tempted to do the work for the student. Instead, I must take time to ask thought-provoking questions that will lead to understanding.

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8. Avoid answering my own questions. Answering my own questions only confuses students because it requires them to guess which questions I really want them to think about, and I want them to think about all my questions. I also avoid rhetorical questions.

9. Ask questions of the whole group. As soon as I direct a question to an individual, I suggest to the rest of the students that they are no longer required to think.

10. Limit the use of group responses. Group responses lower the level of concern and allow some students to hide and not think about my questions.

11. Do not allow students to blurt out answers. A student's blurted out answer is a signal to the rest of the class to stop thinking. Students who develop this habit must realize that they are cheating other students of the right to think about the question.

Summary

Like most teachers, I entered the teaching profession because I care about children. It is only natural for me to want them to be successful, but by merely telling them answers, doing things for them, or showing them shortcuts, I relieve students of their responsibilities and cheat them of the opportunity to make sense of the mathematics that they are learning. To help students engage in real learning, I must ask good questions, allow students to struggle, and place the responsibility for learning directly on their shoulders. I am convinced that children learn in more ways than I know how to teach. By listening to them, I not only give them the opportunity to develop deep understanding but also am able to develop true insights into what they know and how they think.

Making extensive changes in curriculum and instruction is a challenging process. Much can be learned about how children think and learn, from recent publications about learning styles, multiple

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intelligences, and brain research. Also, several reform curriculum projects funded by the National Science Foundation are now available from publishers. The Connected Mathematics Project, Mathematics in Context, and Math Scape, to name a few, artfully address issues of content and pedagogy.

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Multiplication with Fractions: A Piagetian, Constructivist Approach

by Mary Ann Warrington and Constance Kamii

On the basis of Piaget's (1954, 1960) constructivism, Kamii (1989, 1994) has demonstrated that children in the primary grades can invent their own procedures for solving multidigit problems with whole numbers. A significant finding of this research is that when children are not taught algorithms, such as those of "carrying" and "borrowing," their number sense and knowledge of place value are far superior to those of students who have been taught these rules. Warrington (1997) extended this work to the fifth- and sixth-grade level and described an approach to "teaching" division with fractions without teaching the algorithm of "invert and multiply." This article describes a constructivist approach to multiplication with fractions.

In a constructivist class, the teacher does not tell children how to solve any problem but, instead, presents problems and encourages children to invent their own ways of solving them. When the children give answers, the teacher tries not to say that the answer is correct or incorrect. Instead, she or he encourages children to agree or disagree with one another and to exchange ideas until agreement is reached about what makes sense. This approach is radically different from traditional instruction in which the teacher is the one who always knows the answer. In a constructivist classroom, the children are empowered by the knowledge that the answers lie within them. They share strategies and solutions, debate with one another, and think critically about the best way to solve each problem.

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Warrington was the teacher of a self-contained, fifth- and sixth-grade, multiage, mixed-ability class in an independent school near Boston. Prior to entering her class, the children had been exposed to a variety of teaching practices in mathematics. Some had been taught algorithms, some had invented their own procedures for two years, and some had worked extensively with manipulatives. The culture of the classroom as well as that of the school was one that valued the process of learning, and the children were accustomed to exchanging ideas.

The following description of Warrington's teaching and of her students' responses is written in the first person to communicate what she observed and how she made her decisions. We begin with the questions she posed concerning half of whole numbers and of fractions. We then proceed to questions about a third of whole numbers and of fractions and go on to more complicated problems involving nonunit fractions and mixed numbers.

Half of Whole Numbers and of Fractions

Children construct new knowledge out of what they already know. Therefore, when I introduce a new topic, I begin with the familiar and move forward. Knowing that half is the easiest fraction, I began the discussion with the following problem that asked for half of a whole number:

> If Willy had 15 ounces of orange juice in his Thermos, and he drank half of it during lunch, how many ounces of juice did he drink during lunch?

All the students quickly answered this question, and I proceeded to the next one:

> If Willy had drunk one-fourth of 15 ounces during lunch, how much juice would he have drunk?

The children responded rather easily that "cutting" the previous answer "in half" would give the answer of 3 3/4.

> I went on to the next problem involving two fractions, which would usually be solved with multiplication:

> If you had half of an apple pie and you ate half of it, how much pie did you eat?

Many students immediately drew pictures, such as the one in figure 12.1a, but a few replied, "You ate a half." This statement brought a dissenting opinion from someone who explained, "You ate half of a half. You only had half of a pie to begin with; so when you ate half of it, you had a fourth left. So you ate one-fourth." The class agreed that this explanation made sense, and I went on to similar problems involving half of the one-fourth that was left over, and then half of the one-eighth that was later left over. Many children drew the kinds of pictures that can be seen in figures 12.1b and 12.1c.

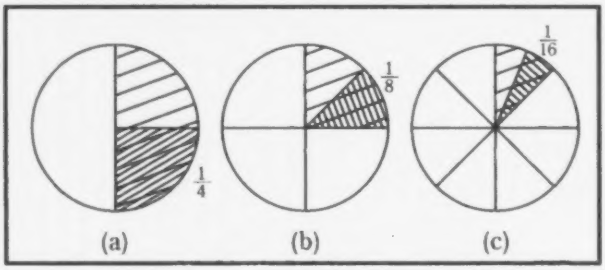

Fig. 12.1. Student drawings of (a) half of 1/2, (b) half of 1/4, and (c) half of 1/8

I then proceeded to nonunit fractions by asking the following question:

> What is half of 3/4?

Three ways were found to solve this problem. The first child called on drew a picture like figure 12.2a and explained, "Three-fourths is the same as 1/4 + 1/4 + 1/4. So I took half of 1/4, which is 1/8, and three times that is 3/8." The next child also broke 3/4 down but thought about it as 1/2 and 1/4 (see fig. 12.2b). He figured out that half of a half is 1/4, which he knew is the same as 2/8. After getting half of 1/4 equals 1/8, he added this amount to 2/8 and got the final answer of 3/8. The third explanation came from a student who decided to "change 3/4 to 6/8 and take half of 6/8, which is 3/8." I was especially impressed with the ease and efficiency of this third method, and everybody agreed that all three methods made sense. The children's ease can be explained by the fact that they were already good at making equivalent fractions in dealing with addition and subtraction problems.

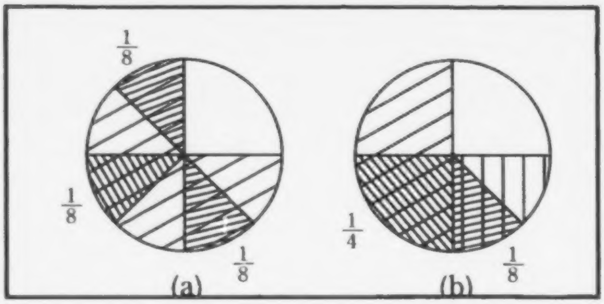

Fig. 12.2. Two ways of thinking about half of 3/4

I decided next to introduce problems involving thirds, sixths, and ninths, which I knew to be more difficult than halves, fourths, and eighths.

A Third of Whole Numbers and of Fractions

As usual, I began with whole numbers, which were easy for the entire class:

> If you had 3 pencils and 1/3 of them were blue, how many blue pencils would you have?

Since this problem was handled with ease, I decided to throw in a nonunit fraction in the next problem:

2/3 of 3

The method for solving this problem was simply to take 1/3 of 3 and to double the answer.

The next question I wrote on the chalkboard was the following:

1/3 of 9

The correct answer of 3 came quickly, and I posed the next problem:

1/3 of 1

The responses did not come immediately this time, and many children drew a pie and split it into three parts. A tentative 1/3 was called out, and everybody agreed after all that this response was correct. I wrote the next problem:

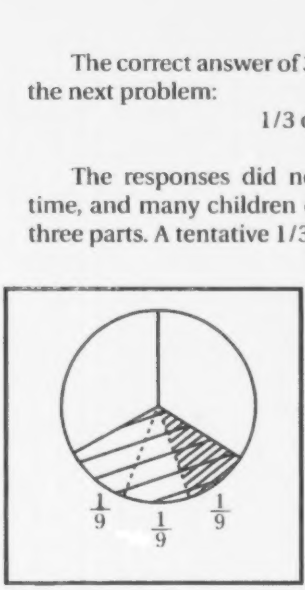

Fig. 12.3. Thinking about one-third of 1/3

1/3 of 1/3

No immediate responses were forthcoming, and so, as usual, I asked the children to estimate the answer. When an answer is not obvious, I always ask the children to estimate before attempting to get an exact answer. The estimates given were 1/9 and "less than 1/6." The explanation of the child who gave the second estimate was "I'm not sure what the answer is, but I know that it is less than 1/6 because half of 1/3 is 1/6, and 1/3 is smaller than a half; so you're taking even less of the third." Despite the fact that this child did not get beyond an estimate, her reasoning was excellent—something that rarely happens in a traditional classroom in which students merely multiply the numerators and denominators. The child who claimed that the answer was 1/9 reasoned that "if you had 1/3 of something and you took 1/3 of it (see fig. 12.3), you break the 1/3 into 3 parts, and each would be 1/9."

More Complicated Problems with Nonunit Fractions and Mixed Numbers

After a few days of working through several more problems that would usually be solved by multiplying fractions, I presented the class with the following question:

> Michael walked 6 5/8 miles to help raise money for a charitable cause. Ariel walked 1/3 the distance that Michael walked. Shanon walked twice as far as Ariel. How far did Shanon walk?

I began by listing the following estimates that the children offered: 4 1/2, 2 1/3, 4 2/3, and "between 4 and 5." The students proceeded to exact computation, and I later listed on the chalkboard the different answers they got: 4 10/24, 4 5/12, and 4 5/24.

When I asked for an explanation of the first answer, the student responded with, "Well, Michael walked 6 5/8 miles, and Ariel walked only 1/3 of that; so first I took 1/3 of 6, which is 2, and then I took 1/3 of 1/8, which is 1/24." I interrupted here and asked her to explain how she knew that. As she drew a picture like the one in figure 12.4, she explained, "If you had only 1/8 of a pie and you take 1/3 of it, you're breaking that small piece into 3 parts, and if you did that to each piece of the pie, there would be 24 pieces; so 1/3 of 1/8 is 1/24." Her peers and I were all satisfied with this explanation, and she continued with, "If 1/3 of 1/8 is 1/24, then 1/3 of 5/8 would be 5/24."

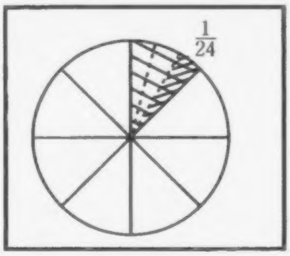

Fig. 12.4. Thinking about one-third of 1/8

Another child interrupted at this point and asked why this outcome was so. The child at the chalkboard first asked, "Do you understand why 1/3 of 1/8 is 1/24? [A nod indicated yes.] Well, 1/3 of 5/8 is 5/24 because if you take 1/3 of 1/8 for five of those pieces of the pie, you would have 5/24." She continued, "Ariel walked 2 5/24 miles, and Shanon walked twice as far; so you double 2 5/24, and you get 4 10/24, which is the same as 4 5/12." This student had initially not reduced her answer but now made the relationship between her answer and another one that was on the chalkboard. At this point, the student who had given the answer 4 5/24 announced that he wanted to change his answer because "I forgot to double the fraction part."

The children continued to tackle other similar problems in similar ways, and I marveled at their ability to invent their own logical ways of solving complex problems. A few days later, I decided to give the following problem to the class:

> Tobin went apple picking and later baked 4 apple pies. He ate 1/5 of one of the pies and put the remaining pies on his back porch to cool. His sister and her friends came along and ate 1/2 of the pies on the porch. How much pie was left on the porch?

The students were used to estimating and promptly gave the estimates of 2, 1 5/10, and "between 1 1/2 and

2." They then set out to find the exact answer. Many children drew pictures, such as figure 12.5, and one child's answer of 1 9/10 was justified in the following way: "First I took 1/5 away from 4 because Tobin ate 1/5 of a pie; so there were 3 and 4/5 left on the porch. His sister ate half of these; so I took half of 3, which is 1 1/2, and half of 4/5, which is 2/5. Then I had to add 1 1/2 and 2/5. I changed 2/5 to 4/10 and [changed] 1/2 to 5/10. Four-tenths plus 5/10 is 9/10; so the answer is 1 and 9/10."

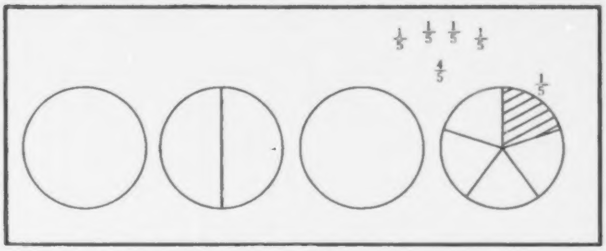

Fig.12.5. Thinking about half of 3 4/5 pies

When I asked if anyone had solved the problem differently, the following procedure was offered: "I started doing it that way [like the previous student], but I didn't want to find half of 3 4/5; so I figured it was close to 4; so I rounded the 3 4/5 up to 4, and I took half of 4, which is 2. But then there was the 1/5 that Tobin ate; so I took half of that, which is 1/10. I subtracted the 1/10 from 2 and got 1 9/10 pies left over." This solution especially impressed me. Children who are encouraged to think for themselves are not limited by algorithms. Answers to problems lie within them, and my respect for their resourcefulness continues to increase.

A week later, I gave the following problem to the class:

> A half of a pound of jelly beans was in a container. Georgia and Emma ate 2/3 of the jelly beans in the container. How much did they eat?

By now, the entire class could estimate the answer as somewhere between 1/2 and 1/3, often by drawing pictures, such as figure 12.6. When they calculated the exact answer, they used three different strategies. The first one was invented by a child who began by taking 1/3 of 1/2, which he said was 1/6 because "1/3 of 1 equals 1/3; so 1/3 of 1/2 is half of that." She then doubled the 1/6 and got 2/6 equals 1/3. The second child who shared his thinking took 2/3 of 1 (equaling 2/3), and by halving 2/3, he got the same answer of 1/3.

The third strategy shared was characteristic of the brilliant efficiency that one can find in a constructivist classroom. The student realized that she had to find 2/3 of 1/2; so she solved 1/2 of 2/3 and immediately got the answer. Many of her classmates were stunned and impressed with the ease of solving the problem this way, but others were skeptical and asked, "Can you do that?" She said, "Yes, it's the same thing because if you take 1/2 of 12, you get 6; and if you take 12 one-halves, you still get 6." Most of the students were convinced that this explanation was reasonable, but a few continued to feel the need to test it by trying it out on other numbers. As a teacher, I was delighted with the exchange of ideas that took place as a result of this problem and the learning that it fostered. But beyond this pleasure, I was fascinated that a student had independently constructed the commutative property of multiplication with fractions, without any formal instruction on multiplication with fractions. This incident was the first time that I thought about the usefulness of this property with respect to solving problems.

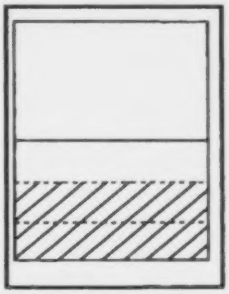

Fig. 12.6. Thinking about two-thirds of 1/2

Conclusion

The reader must have noticed that none of our students invented anything like the algorithm that is found in textbooks. In fact, the preceding problems were not even presented as multiplication problems. When children later encounter a written form, such as "1/2 × 4/5," the teacher tries to encourage them to relate this form to what they have been doing. So she writes

1/2 × 1

on the chalkboard and asks, "What do you think this means?" The usual response is that it means "1/2 one time (equaling 1/2)." Next she writes the following on the chalkboard, asking the class what it means:

1/2 × 2

The students typically say that it means "1/2 two times (equaling 1)." The teacher then writes

1/2 of 2

which is familiar and easy (equaling 1). She may then ask what this expression means:

$$1/2 \times 1/2$$

This one is harder, but someone is likely to say that it means "1/2 half a time (equaling 1/4)." The teacher then writes

$$1/2 \text{ of } 1/2$$

which is familiar (equaling 1/4). If the students are in agreement up to this point, she goes on to write pairs of problems on the chalkboard, such as

$$1/2 \times 2/3 \text{ and } 1/2 \text{ of } 2/3,$$
$$1/2 \times 4/5 \text{ and } 1/2 \text{ of } 4/5.$$

The children soon conclude that the pairs must mean the same thing not only because they always result in the same answers but also because if 1 × 4/5 equals 4/5, 1/2 × 4/5 has to be 2/5. Therefore, "1/2 × 4/5" has to mean "half of 4/5." Likewise, if 1 × 3/9 equals 3/9, 1/3 × 3/9 has to be 1/9. Therefore, "1/3 × 3/9" has to mean "a third of 3/9."

By telling children that "of means to multiply," we impose an arbitrary definition that makes no sense to them. By giving them the algorithm of multiplying the numerators and the denominators, we impose a rule that does not make sense to students. The children in Warrington's class could simply look at 4/5 and know that half of it was 2/5.

Speaking of algorithms, Mack (1990) stated that children's knowledge of algorithms is often faulty and frequently interferes with their thinking in two ways: "(1) [k]nowledge of procedures often kept students from drawing on their informal knowledge of fractions even for problems presented in the context of real-world situations" (p. 27) and (2) "students often trusted answers obtained by applying faulty procedures more than those obtained by drawing on [their own] informal knowledge" (p. 27).

Traditional instruction is based on the outdated assumption that children must internalize in ready-made form the results of centuries of construction by adult mathematicians. Children will go much further, with depth, pleasure, and confidence, if they are allowed to construct their own mathematics that makes sense to them every step of the way.

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The Assessment Principle

ASSESSMENT should support the learning of important mathematics and furnish useful information to both teachers and students.

Assessment should –

- convey a message to students about what kinds of mathematical knowledge and performance are valued;
- provide feedback to students to help them set goals, assume responsibility for learning, and become more independent learners, and also provide examples of complete and correct responses;
- be worthy of students' time and attention and be consistent with the activities used in instruction;
- focus on students' understanding as well as their procedural skills;
- utilize a variety of techniques appropriate to the age, experience, and special needs of students;
- provide information teachers need to make appropriate instructional decisions and build a picture of individual students' progress toward the goalsof instruction. (NCTM 2000, pp. 22–24)

"Assessment should be more than merely a test at the end of instruction to see how students perform under special conditions; rather, it should be an integral part of instruction that informs and guides teachers as they make instructional decisions"(NCTM 2000, p. 22). In addition to highlighting these recommendations, the articles in this group illustrate how assessment methods need to adjust to our changing expectations for student output under *Standards*-based curricula.

The first article presents an example of how students can become partners in the assessment process. Rubrics for self-assessment help students understand the nature of the learning that is valued, as well as monitor their own progress in these areas. Billstein's discussion of the Stem Model outlines a set of principles on which that program's assessment is based and provides an example of how the teacher's rubric can be used to assess the quality of a student's solution to a particular problem.

Cohen and Fowler emphasize the importance of assessing both mathematical understandings and skills. By "assessing it all," students receive the message that a range of competencies is valued—not only basic skills but also mental mathematics, use of appropriate vocabulary, problem-solving abilities, calculator mathematics, and conceptual understanding. The authors suggest appropriate assessment-design strategies, illustrated with test items from middle school–level mathematics.

In the final article, Moskal addresses the problem of how to elicit and understand students' responses to open-ended tasks. The author's careful analysis of four sixth graders' written responses to a problem involving the area of an irregular shape illustrates the information that can be obtained from a careful study of students' work. These examples include both correct answers that result from correct reasoning and correct answers that are based on faulty mathematics, highlighting the importance of basing assessment on more than just the final answer to a problem.

The Stem Model

by Rick Billstein

THE SIX Through Eight Mathematics (STEM) project is one of the National Science Foundation's (NSF) funded middle school mathematics curriculum projects. The project received initial funding in 1992; the finished product is expected to be available in 1998. The STEM curriculum is organized in teaching units called modules. Each module is organized around a conceptual theme, such as "flight" or "wonders of the world." The use of thematic modules promotes students' developing and applying mathematical concepts in broad contextual situations so that they do not view mathematics as a collection of separate and unrelated topics.

To raise the level of students' performance in problem solving and communication along with performance in mathematical content, STEM designed an assessment package on the basis of the following principles.

- Assessment should have the improvement of learning as its primary goal. The results of student assessment should inform teachers about the next steps in the teaching process. Information about students' strengths and weaknesses can prevent needless repetition and help identify which concepts have been learned.
- Documenting students' achievement should be an integral part of the instructional process, not an add-on to it. Assessment information can be drawn from instructional tasks, and additional instruction can be given in the context of assessment activities.
- Assessment should include the active participation of students in open problems. Open problems can be open-ended or open-response problems. An open-ended problem has many solutions and many ways to reach a solution. (See fig. 13.1.) An open-response problem has one answer but may have many ways to reach a solution. (See fig. 13.2.) Students should be involved with nonroutine

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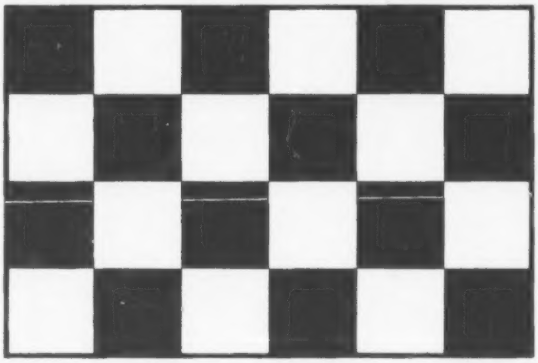
Design an ideal chair for students at your grade level. Be sure to tell how you obtained the data on which you based your design.

Fig. 13.1. Example of an open-ended problem



How many different squares do you see in this sports flag? Describe the method you used to find all the squares. Will your method work for any size checkered flag?

Fig. 13.2. Example of an open-response problem

open problems of both kinds, and alternative approaches to solutions should be encouraged and discussed.

- Assessment should reflect real-world applications. Students should be exposed to many real-world applications. Students should be able to specify the problem, collect and organize appropriate data, make conjectures from the data, formulate an appropriate model, validate and refine the model, and communicate the results in written and oral form.

- Assessment should permit the full use of technology. Assessments that truly assess mathematical power and not just computational skill should allow students to use all appropriate tools.

- Assessment should use a variety of methods. Assessment should use various methods and not be limited to simplified methods like multiple-choice items. Assessment should measure what is valued in the curriculum, not just what is easy to measure. The emphasis should be on assessing what students know and how they think about mathematics rather than what they do not know.

Although STEM uses several assessment techniques, including summative content activities, student journals, embedded assessments, and portfolios, this article focuses on the use of open questions and their role in assessment. These types of questions involve students in activities that require interaction, communication, and decision making in nonroutine situations. Open questions often reveal more about students and their thinking than do traditional questions. Open questions can be found in the project materials in two places: as part of extended explorations (E2) and embedded within the instructional materials and practice exercises. Many of these problems have real-world applications. Alternative approaches are encouraged and discussed. The solution to these problems involves constructing a mathematical model for the situation. To find a solution, students must define the problem, devise and carry out a plan for solving it, and prepare a report that explains and interprets their solution. Full use of technology is permitted on these explorations. The presentation of the solution can take various forms: a written or oral report, a physical model or display, a dramatic presentation or a news report, or other forms that students can design.

Extended explorations (E2) are investigations or activities on which students work outside of class over a one- to two-week period. The solutions to the E2's are evaluated with two sets of assessment criteria that are written in a language that students can understand. These criteria, or rubrics, are predetermined project standards that cover several levels of student achievement. These standards are based on NCTM's curriculum and evaluation standards (1989). One set is for student self-assessment, a process that has not been a part of a traditional program. Using the self-assessment rubric, students take an active role in assessing their own work, which is essential to improving their learning processes. Note that the self-assessment criteria in figure 13.3a are written from the students' point of view, for example, "I used appropriate mathematical language. . . . "

A similar set of teachers' assessment criteria, shown in figure 13.3b, is used to assess students' work from a teacher's point of view, for example, "You used appropriate mathematical language. . . ."

The combination of the student- and teacher-assessment criteria provides feedback so that students can improve as they work through the curriculum. As students become familiar with the assessment criteria, they understand what is needed to improve their problem-solving and communication skills. The assessment criteria allow the teachers to recognize a student's strengths and help the student focus on areas where growth is needed.

Students are asked to write their solutions to open problems by using mathematical language and their best representations to communicate how they solved the problem; the decisions they made as they solved the problem; and the connections they made to other problems, mathematics, or other subjects. Students' work is assessed according to the following five criteria:

- Problem solving
- Mathematical language
- Representations
- Connections
- Presentation

Through self-assessment and teacher-assessment criteria, students learn exactly what is important in good problem solving and communication and can evaluate their own progress in these areas. Students will respond if they know what is expected and if the criteria

Name ______________________ Problem ______________________

STUDENT SELF-ASSESSMENT

☐ If your score is in the shaded area, explain why on the back of this sheet and stop.

☆ The star indicates that you excelled in some way.

Problem Solving

❶	❷	❸	❹	❺ ☆
I did not understand the problem well enough to get started, or I did not show any work.		I understood the problem well enough to make a plan and to work toward a solution.		I made a plan, I used it to solve the problem, and I verified my solution.

Mathematical Language

❶	❷	❸	❹	❺ ☆
I did not use any mathematical vocabulary or symbols, or I did not use them correctly, or my use was not appropriate.		I used appropriate mathematical language, but the way it was used was not always correct, or other terms and symbols were needed.		I used mathematical language that was correct and appropriate to make my meaning clear.

Representations

❶	❷	❸	❹	❺ ☆
I did not use any representations, such as equations, tables, graphs, or diagrams, to help solve the problem or explain my solution.		I made appropriate representations to help solve the problem or help me explain my solution, but they were not always correct or other representations were needed.		I used appropriate and correct representations to solve the problem or explain my solution.

Connections

❶	❷	❸	❹	❺ ☆
I attempted or solved the problem and then stopped.		I found patterns and used them to extend the solution to other cases, or I recognized that this problem relates to other problems, mathematical ideas, or applications.		I extended the ideas in the solution to the general case, or I showed how this problem relates to other problems, mathematical ideas, or applications.

Presentation

❶	❷	❸	❹	❺ ☆
The presentation of my solution and reasoning is unclear to others.		The presentation of my solution and reasoning is clear in most places, but others may have trouble understanding parts of it.		The presentation of my solution and reasoning is clear and can be understood by others.

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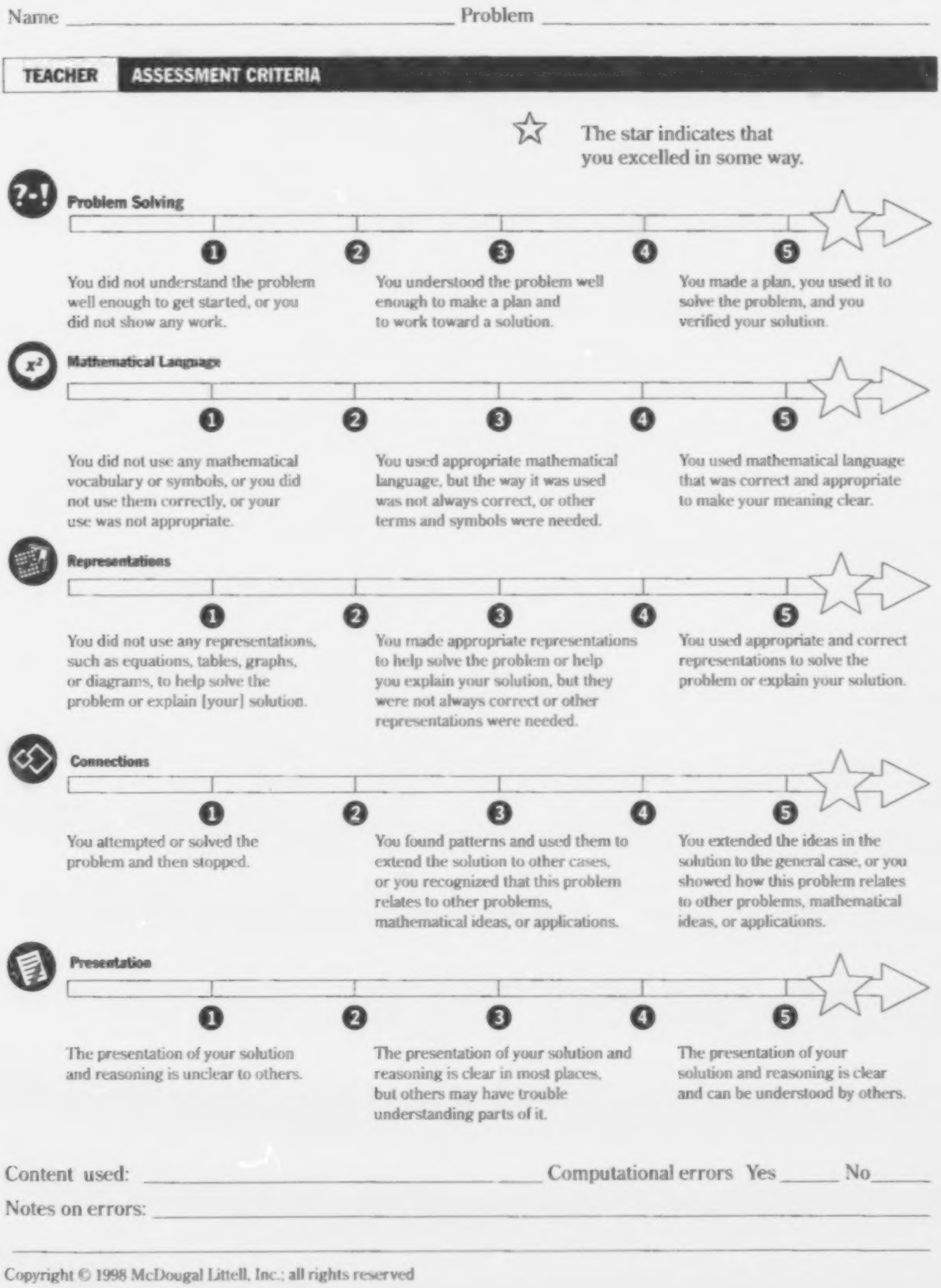
Fig.13a. STEM student self-assessment criteria



Name ______________________ Problem ______________________

TEACHER **ASSESSMENT CRITERIA**

☆ The star indicates that you excelled in some way.

Problem Solving

❶ ❷ ❸ ❹ ❺ ☆

You did not understand the problem well enough to get started, or you did not show any work.

You understood the problem well enough to make a plan and to work toward a solution.

You made a plan, you used it to solve the problem, and you verified your solution.

Mathematical Language

❶ ❷ ❸ ❹ ❺ ☆

You did not use any mathematical vocabulary or symbols, or you did not use them correctly, or your use was not appropriate.

You used appropriate mathematical language, but the way it was used was not always correct, or other terms and symbols were needed.

You used mathematical language that was correct and appropriate to make your meaning clear.

Representations

❶ ❷ ❸ ❹ ❺ ☆

You did not use any representations, such as equations, tables, graphs, or diagrams, to help solve the problem or explain [your] solution.

You made appropriate representations to help solve the problem or help you explain your solution, but they were not always correct or other representations were needed.

You used appropriate and correct representations to solve the problem or explain your solution.

Connections

❶ ❷ ❸ ❹ ❺ ☆

You attempted or solved the problem and then stopped.

You found patterns and used them to extend the solution to other cases, or you recognized that this problem relates to other problems, mathematical ideas, or applications.

You extended the ideas in the solution to the general case, or you showed how this problem relates to other problems, mathematical ideas, or applications.

Presentation

❶ ❷ ❸ ❹ ❺ ☆

The presentation of your solution and reasoning is unclear to others.

The presentation of your solution and reasoning is clear in most places, but others may have trouble understanding parts of it.

The presentation of your solution and reasoning is clear and can be understood by others.

Content used: ______________________ Computational errors Yes _____ No_____

Notes on errors: ______________________

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Fig.13.3b. STEM teacher assessment criteria

make the expectations very clear. These same assessment criteria are used in all three years of the project. They are introduced in the first module in grade 6 and are reviewed in the first module of each subsequent grade level. These assessment criteria are not add-ons to the curriculum but are an integral part of the teaching materials. The criteria clearly communicate to the students what is valued in the curriculum.

In the prepilot version of the materials, the criteria were not user-friendly and were hard to interpret. On the basis of teacher feedback, the rubrics have undergone many revisions before reaching their present forms. Notice the similarity between the criteria and the first four of the NCTM's (1989) curriculum and evaluation standards: problem solving, communication, reasoning, and connections. The assessment criteria show students that these standards are valued and that they are an important part of the curriculum. Cheryl Wilson, a former middle school mathematics teacher turned principal, reports on her experience with the STEM assessment criteria:

> This year I have used drafts of this tool in my classroom and was amazed at how quickly students improved in their ability to solve complex problems and to write up their problem-solving solutions. This is a tool whose time has come. Assessment has been the focus of several recent professional journals, conferences, and discussions. It is exciting to see a curriculum project develop a quality assessment instrument to accompany the curriculum materials.

The assessment criteria are not mastered immediately by either the teacher or the students. Mastery takes time and is part of the learning process. Peer assessment is also used to help students become comfortable with the criteria. The more that teachers and students work with the criteria, the more comfortable they become using them. Learning to work with these criteria can be thought of in the same way as learning a new language. At the beginning, communicating is very hard, but before long, teachers and students alike are using the same vocabulary and looking for the same things as they assess solutions to problems.

The assessment criteria are normally completed with a marking pen, and the score on each scale is marked along the ray below each heading. Notice the star after the 5 on each criterion. This feature was added to the original rubric so that if a student really excelled in some way, the effort could be rewarded. Students strive all year to score at this highest level, but it is reserved for exceptional performances.

To better illustrate how the assessment criteria work, a sample response on the E2 in figure 13.4 from a student named Marge is given in figure 13.5. A discussion of Marge's scores follows.

E2

Russian Peasant Sets World Record in 1700's

The greatest officially recorded number of children born to one mother is 69. In 27 different pregnancies she gave birth to 16 pairs of twins, 7 sets of triplets, and 4 sets of quadruplets.

What is the probability of having a family of one sex if the family consists of—

a. one child?
b. two children?
c. three children?
d. four children?
e. five children?

Fig. 13.4

Problem solving. Marge gave a level-5 response here. She used charts and tables to find a pattern and worked toward finding a general rule. Her explanations about her decisions as she solved the problem were clear, and her reasoning was correct. Her approach worked well, and she solved the problem.

Mathematical language. Marge correctly used some mathematical language, such as probabilities, patterns, and general rule, and she used correct notations for fractions, percents, and exponents. However, she should have referred to the "bottom number" as the denominator. Also, 2^n is not n raised to a power of 2, but rather 2 is a *base* and n is an *exponent*, or *power*. Because she could have used more mathematical language and some language was not used correctly, this response is at level 3.

Representations. Marge's representations, a table, charts, and an equation, are accurate and appropriate for the problem, for a level-5 response.

Connections. Marge arrived at an algebraic generalization that showed how to find the probability for any number of children, which is a level-5 response. If she

Probabilities of the Same Sex

I wanted to find the probabilities of families having all girls or all boys if the families had 1 child, 2 children, 3 children, 4 children, and 5 children. Maybe I could find out the probability of the Russian peasant's having all boys or all girls.

I decided to look at a family of 1 child first, then 2 children, and so on, and see if I could find a pattern. I made the charts below to find the probabilities. The charts show all the ways the births could happen. I only made a chart for 1, 2, and 3 children because I found a pattern.

#	1	1	
B	✓		
G		✓	

1 child

	1		2		3		4	
#	1	2	1	2	1	2	1	2
B	✓	✓	✓					✓
G				✓	✓	✓	✓	

2 children

	1			2			3			4			5			6			7			8		
#	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
B	✓	✓	✓	✓	✓		✓						✓		✓		✓			✓	✓			✓
G						✓		✓	✓	✓	✓	✓		✓		✓		✓	✓			✓	✓	

3 children

I made a table showing the probabilities of 1 sex for families with different numbers of children.

# of children	Probabilities of 1 sex
1	$\frac{2}{2}$, or 100%
2	$\frac{2}{4}$, or 50%
3	$\frac{2}{8}$, or 25%
4	$\frac{2}{16}$, or 12.5%
5	$\frac{2}{32}$, or 6.25%

The pattern is that the probabilities get smaller by half for every extra kid. This pattern gave me the data, and it was much quicker. I could get a general rule:

If n = # of children, then the probability of the same sex is $\frac{2}{2^n}$.

I figured out the formula because I saw that the bottom number doubled with each extra kid. I figured that if the number of children was raised to the power of 2, the probability would always be given by $\frac{2}{2^n}$.

If there were 3 kids, then the probability of the same sex is $\frac{2}{2^3} = \frac{2}{8}$. I checked that this is what I got in my chart. I checked it for all the other cases.

I think that the probability that all the children of the Russian woman being the same sex is $\frac{2}{2^{69}}$, which I think is a very small number, so it probably would not happen.

Fig.13. 5. Marge's work

had extended her solution to more cases than called for, such as for ten children, this response would have been at level 4.

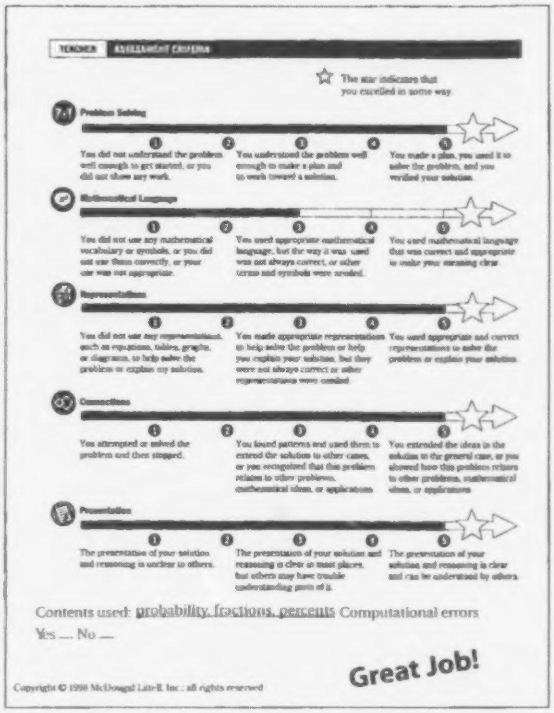
Presentation. Marge's presentation is clear and can be understood by others—a level-5 response. Note that she did not have to type the material or produce computer graphics to score at level 5.

Marge would receive a teacher-assessment criteria that looks like the one in figure 13.6. If a student's self-assessment and the teacher's assessment do not agree, a great opportunity arises for a discussion between the student and the teacher. These discussions can lead to improved problem-solving and communication skills. If students assess themselves on a regular basis, they will become more accustomed to the criteria. Students know from the beginning what is important and what they have to do to score well. If one or more areas need work, a glance at the returned rubric tells the student immediately where effort is needed to improve performance.

Not all problems elicit all the criteria. Sometimes students are not scored on a particular criterion because it is not appropriate for the problem. In this situation, the criterion that is ignored is not counted when determining a grade on the activity. Profiles are available for those teachers who want to translate the rubric to a letter grade or a score to record in their grade book. A sample of a "B" profile is given in figure 13.7.

Teachers and students found that it took time and work to become comfortable using the criteria. Some sample comments from field-test teachers follow:

> For my students, I think that this gives them real honor. It gives them real pride. It worked really well for me. It probably took me a good third of the year before they understood how to do it. When they left, everybody was a better evaluator.
>
> I think it is great. The kids really loved it. They liked the self-assessment. Amazingly, they were honest with their opinions about what they had done.
>
> This can be very positive for the kids because they don't have to do it right. If they have graphs and tables and are communicating to express things using mathematical terms, hey, that's very good. It's good at building self-esteem at this point.
>
> It's excellent. The criteria that are written underneath each of the numbers are very specific but able to encompass lots of different problems. . . . It was not



TEACHER ASSESSMENT CRITERIA

The star indicates that you excelled in some way.

Problem Solving

① You did not understand the problem well enough to get started, or you did not show any work.
②
③ You understood the problem well enough to make a plan and to work toward a solution.
④
⑤ You made a plan, you used it to solve the problem, and you verified your solution.

Mathematical Language

① You did not use any mathematical vocabulary or symbols, or you did not use them correctly, or your use was not appropriate.
②
③ You used appropriate mathematical language, but the way it was used was not always correct, or other terms and symbols were needed.
④
⑤ You used mathematical language that was correct and appropriate to make your meaning clear.

Representations

① You did not use any representations, such as equations, tables, graphs, or diagrams, to help solve the problem or explain my solution.
②
③ You made appropriate representations to help solve the problem or help you explain your solution, but they were not always correct or other representations were needed.
④
⑤ You used appropriate and correct representations to solve the problem or explain your solution.

Connections

① You attempted or solved the problem and then stopped.
②
③ You found patterns and used them to extend the solution to other cases, or you recognized that this problem relates to other problems, mathematical ideas, or applications.
④
⑤ You extended the ideas in the solution to the general case, or you showed how this problem relates to other problems, mathematical ideas, or applications.

Presentation

① The presentation of your solution and reasoning is unclear to others.
②
③ The presentation of your solution and reasoning is clear in most places, but others may have trouble understanding parts of it.
④
⑤ The presentation of your solution and reasoning is clear and can be understood by others.

Contents used: probability, fractions, percents Computational errors Yes __ No __

Great Job!

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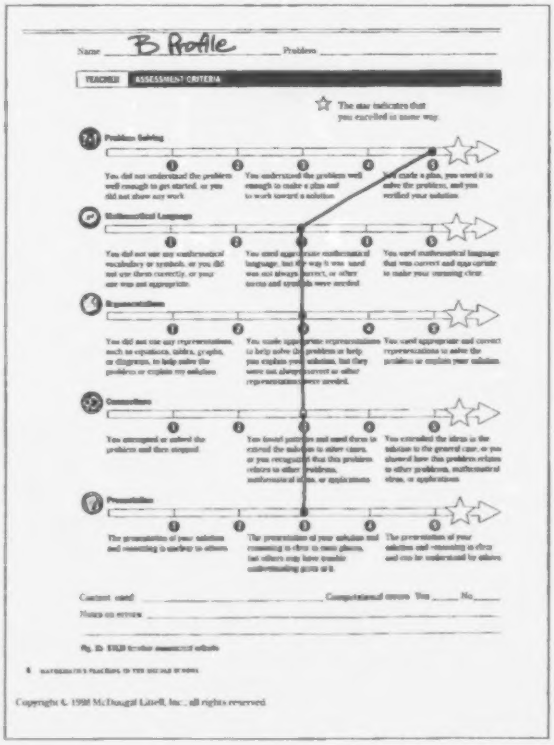
Fig. 13.6. Teacher assessment of Marge's work



Name B Profile Problem ______

TEACHER ASSESSMENT CRITERIA

The star indicates that you excelled in some way.

Problem Solving

① You did not understand the problem well enough to get started, or you did not show any work.
②
③ You understood the problem well enough to make a plan and to work toward a solution.
④
⑤ You made a plan, you used it to solve the problem, and you verified your solution.

Mathematical Language

① You did not use any mathematical vocabulary or symbols, or you did not use them correctly, or your use was not appropriate.
②
③ You used appropriate mathematical language, but the way it was used was not always correct, or other terms and symbols were needed.
④
⑤ You used mathematical language that was correct and appropriate to make your meaning clear.

Representations

① You did not use any representations, such as equations, tables, graphs, or diagrams, to help solve the problem or explain my solution.
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③ You made appropriate representations to help solve the problem or help you explain your solution, but they were not always correct or other representations were needed.
④
⑤ You used appropriate and correct representations to solve the problem or explain your solution.

Connections

① You attempted or solved the problem and then stopped.
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③ You found patterns and used them to extend the solution to other cases, or you recognized that this problem relates to other problems, mathematical ideas, or applications.
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⑤ You extended the ideas in the solution to the general case, or you showed how this problem relates to other problems, mathematical ideas, or applications.

Presentation

① The presentation of your solution and reasoning is unclear to others.
②
③ The presentation of your solution and reasoning is clear in most places, but others may have trouble understanding parts of it.
④
⑤ The presentation of your solution and reasoning is clear and can be understood by others.

Content used ______ Computational errors Yes ____ No ____

Notes on errors ______

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Fig. 13.7. Possible "B" profile

> easy for them [students] at the beginning. It was my lack of self-assurance. I am used to a red answer key. As I got more confident, students were better able to tell what I expected.

Some sample student comments about what they thought of using rubrics are given next.

> I like having the rubrics because they help me a lot. When I just receive a grade, I don't know why. With the rubric I know why I got the grade and what I can do about it next time.
>
> I like having a rubric before I write something so I know what is expected of me. Sometimes I don't like rubrics because they make me feel a lot is expected of me, but that helps me set higher goals.
>
> Almost every time a rubric has helped me because when I get a grade that I don't like, I can either make it right or I can improve on the next problem.

In the STEM curriculum we have asked students to become involved in real mathematics—to make conjectures, to solve problems, and to present their solutions in a clear manner using convincing arguments. We have tried to design our curriculum around the ideas stated in *Measuring What Counts* (MSEB 1993):

> Important mathematics must shape and define the content of assessment. Appropriate tasks emphasize connections within mathematics, embed mathematics in relevant external contexts, require students to communicate clearly their mathematical thinking, and promote facility in solving nonroutine problems. Considerations of connections, communication, and nonroutine problems raise many thorny issues that testmakers and teachers are only beginning to explore. However, these considerations are essential if students are to meet the new expectations of mathematics education standards.

The use of extended problems and projects and their scoring with generalized multidimensional scoring rubrics are important components in the STEM project. When assessment is improved, learning is improved. The assessment techniques that are used not only indicate to students what is valued but also give them the criteria for judging their achievement. The rubrics engage students in thinking about performance expectations. Students' solutions give valuable information about their conceptual understanding or lack of understanding of mathematical concepts—teachable moments. A good assessment program can provide a mechanism for creating more teachable moments.

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Create Assessments That Do It All

by Ira S. Cohen and Joan V. Fowler

> I learned how to do math that involves some thinking instead of seeing a bunch of numbers to add or subtract.—Elizabeth, grade 8

> I learned from these tests different skills and different techniques in solving many math problems that made it easy and made it seem less complicated.—Michael, grade 8

> I learned from these tests different methods for solving math and understanding math concepts. These tests helped me understand math better because I now know different ways I can solve math problems and know different methods I can use.—Natalie, grade 7

How would you like to hear such comments about your mathematics tests from students? In this article, ideas for an assessment system that reflects NCTM's *Curriculum Standards* (NCTM 1989) are presented. In this system, assessment tasks are created to support instruction that values a mastery of the "basics," mental mathematics, mathematics vocabulary, problem solving and writing in mathematics, calculator mathematics, and mathematics understanding. The motivation behind each task is the need to measure both a student's knowledge and his or her depth of understanding. In addition, as the students have said in their own words, these assessments are intended to be a learning experience. The assessments presented also adhere to the principle that "if it is important, assess it."

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The "Basics"

We start by discussing the basics. Although mathematics educators everywhere strive to shift the focus of their instruction to a greater emphasis on problem solving and critical thinking, most agree that proficiency in certain basic arithmetic skills should not be sacrificed. A classroom assessment system should include assessing the basics, such as being able to perform the following tasks:

- $7n = 28,\ n =$ ____
- $7840 \div 28$
- $0.84 \div 0.16$
- $12.0603 + 0.24 + 4.2$
- Find the least common multiple for the numbers 2, 5, and 6.
- $2^3 + 6 - (2 \times 4) + 7$
- $3\frac{5}{6} \times 2\frac{1}{4}$

Mental Mathematics

> These tests were most helpful to me in the area of mental math because it made me see the question in my head, and I had to use strategies to get the answer.—Alexandra M., grade 8

> These tests were most helpful to me in the area of mental math because I am a more visual learner than an auditory learner, and [they] sharpened my skills.—Alexandra A., grade 8

NCTM's curriculum standards emphasize mental mathematics. Too often, however, students consider a topic important only if they know that it will be assessed. As a result, teachers should keep in mind that it is important not just to practice mental mathematics in class but to assess it as well. In addition to improving mental-mathematics skills, making mental-mathematics assessment a regular part of classroom instruction helps students recognize that mentally solving certain problems is more practical than using paper and pencil or a calculator. Many mathematics teachers have been frustrated by middle school students who turn to their calculators to compute 9 × 7, 1/2 of 8, 10 percent of 50, or \$1.99 + \$1.99.

Mathematics educators strive to make mathematics in the classroom as close as possible to mathematics in "real life." Mental mathematics is most definitely a real-life experience. Students who have improved their mental-mathematics skills through their experience with mental-mathematics assessment often, for example, help their parents by computing a percent discount or figuring the tip on a restaurant check.

A mental-mathematics section on an assessment forces a student to employ a series of mental-mathematics strategies. For instance, in the problem 74 ÷ 100, the strategy is that dividing by 100 involves moving the decimal point two places to the left. In the past, many students would use long division to solve this problem. From our experience, we have seen an increase in the number of correct responses. Students do not ask whether a test contains a mental-mathematics section; they have come to expect it. The following problems are easily solved using mental-mathematics strategies only:

- 74×100
- 9×8
- 50% of 30
- 60×40
- $299 + 299$
- $74 \div 100$
- $\frac{1}{5} + \frac{2}{5}$
- $0.64 \div 0.8$
- 50% of 100
- $\frac{7}{8} - \frac{3}{8}$

Mathematics Vocabulary

For students to become proficient in the vocabulary of mathematics, they must use the appropriate terms and symbols and become comfortable with their use. Such questions as the following help our students achieve this goal.

- Use these numbers to answer the following questions:

  8 72 5.9 47 39.89 45.8 100 66.087 0.0765

  — Find two factors that when multiplied result in a product whose last digit is in the tenths place.
  — Find two factors that when multipled result in a product that is > 1.

- Using these numbers, answer the following questions:

  6 9 12 78 26 54 15 24 27 45

  — Use two numbers to write a fraction that is >1.
  — Use two numbers to write a fraction that is < 1.
  — Using the symbol ÷, write a division problem that results in a quotient of 3.

- Using these numbers, perform the following tasks:

  $$\frac{7}{8} \quad \frac{2}{5} \quad \frac{1}{3} \quad \frac{19}{38} \quad \frac{1}{4} \quad \frac{7}{10}$$

  — Write a multiplication problem that results in a product that is > 1. Justify your conclusion.
  — Using two of the fractions, write either a division or a multiplication problem that results in an improper fraction. Give reasons for your choice.

In addition to a knowledge of the mathematics vocabulary being tested, these questions require the use of reasoning and problem-solving skills. If students are not accustomed to giving reasons, they must be reminded to do so. Students need to be given instruction that enables them to accomplish this goal.

Problem Solving and Writing

Another feature of the assessment involves problem solving and writing in mathematics. NCTM stresses that communication and problem solving are important, and students improve in these areas with increased practice. Sample questions that assess problem solving and writing to explain a solution follow. Some of these questions can be used as an in-class learning experience rather than as a means of assessment.

> Bobby was asked to solve the following problem: 67 ÷ 55. His solution was 0.82. Without recalculating the answer, decide whether Bobby's solution is logical. In paragraph form, explain why you agree or disagree with his solution.

Students gave two solutions for this question. One solution changed the problem to the standard division form. The student stated that 55 goes into 67 once and something. Therefore, Bobby's solution is not correct. In another solution, a student said that changing 67 ÷ 55 into a fraction gives 67 in the numerator and 55 in the denominator. Since the number in the numerator is greater than the number in the denominator, the value of the fraction is greater than 1; therefore, Bobby's solution is illogical. Not only were the students asked to write in mathematics, but their understanding of mathematical concepts was assessed through this question.

For homework, the students had to answer the following questions.

1. One hundred fifty students are going on a trip. If a bus holds 45 students, how many buses will they need? Five students gave the following solutions. Brad's solution was 3, Tara's was 3 R 15, Paul's was 3 1/3, Calvin's was 3 15/45, and Hobbes's was 4. (1) Select two solutions, and explain how the students arrived at their answers. (2) Only one solution is correct. Which one? Explain your answer.

2. Sally wants to play a game of chance. In the game, you will roll a single die. Sally will roll a pair of dice. Here are the rules:

- You and Sally will roll the dice at the same time.
- If you roll a 5 on your single die, but Sally does not roll a sum of 7 on her pair of dice, you win.
- If Sally rolls a sum of 7 on her pair of dice, but you do not roll a 5 on your single die, Sally wins.
- If you roll a 5, and Sally rolls her sum of 7, neither one wins and both roll again.
- If you do not roll a 5, and Sally does not roll her sum of 7, roll again.
- You both keep playing until one of you wins.

See the following examples:

Game 1:

ROLLS	YOU (SINGLE DIE)	SALLY (PAIR OF DICE)
1	4	8
2	3	7 Sally wins.

Game 2:

ROLLS	YOU (SINGLE DIE)	SALLY (PAIR OF DICE)
1	3	10
2	1	6
3	5 You win.	2

Do you think you will win? Explain in writing, with appropriate mathematical examples, why you think you would or would not win. Would you take the bet if the rules of the game changed so that Sally needed a sum of 3 on her pair of dice?

Calculator Mathematics

Assessment should include calculator use in relevant situations. As with their use of mental mathematics, students need to understand when a calculator is an appropriate tool and know how to operate a calculator's keys and functions properly. Questions such as the following encourage the appropriate application of a calculator.

1. The Electrical Manufacturing Company had a very successful year. The following chart shows the amount of sales for each month for the calendar year 1997. The president of the company asked his finance advisor to determine the monthly average amount of sales in 1997.

Month	Sales
January	$1,535,868
February	$1,324,643
March	$909,986
April	$674,124
May	$1,563,235
June	$890,098
July	$750,001
August	$345,641
September	$545,221
October	$1,428,086
November	$672,134
December	$1,654,235

2. The heart pumps about 200 liters of blood in one hour. About how many liters will your heart pump in a year? Show the numbers that you are using to solve this problem.

3. Complete the following:

2 × 9 = ______ 3 × 9 = ______
4 × 9 = ______ 5 × 9 = ______
6 × 9 = ______ 7 × 9 = ______
8 × 9 = ______ 9 × 9= ______

4. Use your calculator to determine 2 × 99, 3 × 99, 4 × 99, . . . , 9 × 99 and to determine 2 × 999, 3 × 999, 4 × 999, . . . , 9 × 999. Describe the pattern that you observe. Using this pattern, predict the values of 5 × 9,999 and 6 × 99,999. Check your predictions with your calculator.

Mathematics Understanding

Questions that assess mathematics understanding go beyond mathematics. They target a student's depth of understanding of the concept being tested. For example, when we ask a student to compute 3.8 × 4.4 and the student answers correctly with 16.72, we know that the student understands the mechanics of multiplying decimals. We do not know whether the student understands the concept behind the placement of the decimal point in the product. By modifying the question to "Given that 3.8 × 4.4 is 16.72, explain how you could find the product of 0.38 × 0.44 without recalculating," we learn by students' responses the depth of their conceptual understanding of multiplying decimals. Sample questions follow.

1. Andy was faced with the following problem. He was given these four fractions and asked to put the fractions in numerical order from the least to the greatest. Using any two mathematical techniques, explain, in paragraph form, how you solved Andy's problem.

   $$\frac{5}{9} \quad \frac{1}{2} \quad \frac{3}{27} \quad \frac{4}{6}$$

2. Choose 2 of these numbers to perform the following:

   $$\frac{7}{8} \quad \frac{2}{5} \quad \frac{1}{3} \quad \frac{19}{38} \quad \frac{1}{4} \quad \frac{7}{10}$$

   Write a division problem that results in a quotient that is > 1. Give reasons for your choice.

Our experience indicates that students have a better understanding of mathematics concepts taught from these assessments than from the traditional tests of twenty-five problems. Students need to be "taught" new methods to be successful on these tests, so instructional methods must be changed. As each new topic is presented to students, assessment tasks in each area must be created and lessons developed to help the students achieve success. Clearly, the assessment objectives inform instruction.

Assessment-Design Strategies

Creating these assessment tasks requires changing the way we typically think about testing mathematics. Often, typical mathematics test questions assess what students can "do." Not enough emphasis is placed on what students think and understand nor on what they can investigate, discuss, and explain about mathematics.

The following strategies can be used to design thought-provoking assessment questions. We suggest that assessment tasks be created with this overall question in mind: What should students understand about the mathematics?

- Ask which problems in the unit being assessed are typically, in real life, done with mental mathematics.
- Make up a problem with the wrong answer. Have students write an explanation of what is wrong and why.
- Have students contrast different solutions to the same problem.
- Redesign questions so that they have more than one solution.
- Give a solution, and have students write the problem.
- Look at all situations encountered in real life as possible mathematics problems.
- Pose questions that require students to understand and explain the relationships among mathematical concepts.
- Pose questions for which students must validate whether a solution is reasonable.
- Create tasks that require students to determine their number sense without computation.

Success on a mathematics test in the past often meant that the students could "do" mathematics. It has been our experience that success on the assessments that we have designed means that the students can "do" and "understand" mathematics.

> I learned all of the different ways you can answer questions. These questions got me thinking even though I did not like doing these problems at the time. I think that they have helped me understand math.—Michelle, grade 7

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Understanding Student Responses to Open-Ended Tasks

by Barbara M. Moskal

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Communicating mathematical knowledge is a challenge for students, and assisting students in clearly expressing their mathematical ideas is a challenge for teachers. Open-ended tasks give students opportunities to select their own approaches for both solving problems and expressing mathematical ideas (Billstein 1998; Conway 1999). Students' responses to these tasks give teachers evidence of their students' problem-solving and communication skills.

This article discusses examples of the detailed explanations that students offered in response to a written, open-ended geometry task, especially the ways in which students communicated their knowledge. As this article illustrates, different students may select different methods of communication, such as using text, diagrams, or mathematical symbols, to display their solution processes. Examining the student solution processes helps the teacher better understand the students' mathematical knowledge.

Student Responses

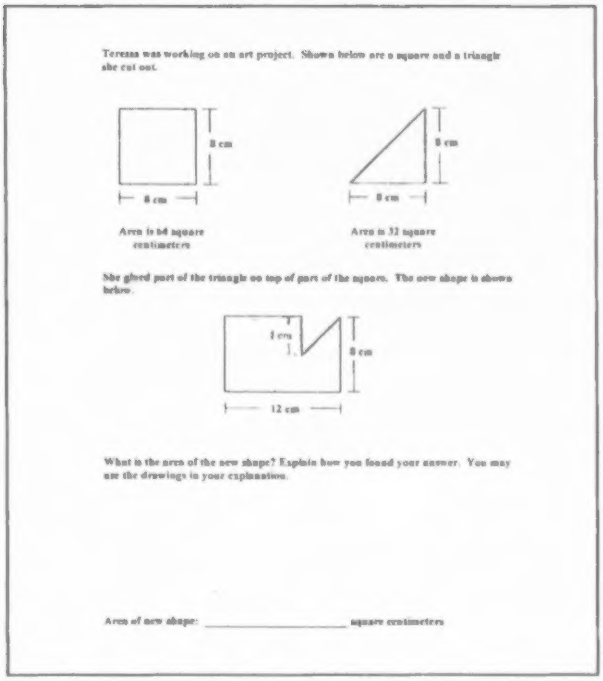
The Irregular Area Task, shown in figure 15.1, was administered to sixth-grade students (Moskal 1997a, 1997b) after they had received instruction in determining the area of squares, rectangles, and triangles. The Irregular Area Task was originally developed as part of the Quantitative Understanding: Amplifying Student Achievement and Reasoning (QUASAR) project (see Lane et al. [1995]). Although a classroom of students completed this task, only four responses will be discussed here. To maintain confidentiality, both the students' and the teacher's names have been changed.

At the start of the area unit, Ms. Harding showed her students how to subdivide a rectangular figure into unit squares. She explained to them that the total number of unit squares that made up the rectangle was the area of the rectangle. Building from this conceptualization of area, she derived with her students the standard formula for finding the area of a rectangle, that is, base × height. She used a similar approach to develop the formula for finding the area of a triangle. Harding dedicated a great deal of classroom attention to explaining how portions of unit squares could be combined to create complete unit squares when finding the area of a triangular figure. Before administering the Irregular Area Task, however, Harding had not addressed how the area concept could be extended to find the area of figures other than squares, rectangles, and triangles. Harding had previously used open-ended tasks in the classroom, and she repeatedly encouraged her students to write complete solutions to open-ended mathematics problems.

Naser's response

Naser, whose response is shown in figure 15.2, used a combination of text and diagrams to explain his answer. His explanation suggests that Naser was attempting to reshape the irregular figure into a large rectangle with a base of 12 cm and a height of 8 cm. He indicated that he had calculated the area of the larger rectangle, but he did not include the outcome of this computation.

Although Naser's overall approach is correct, reforming the figure into a rectangle and removing the area of the missing triangle, he made several mistakes in the solution process. Then he explained that he had subtracted 15 square units from the rectangle to get his answer of 81. Fifteen square units was his calculated area for the missing triangle in the rectangle. Naser had drawn three vertical lines and four horizontal lines, partitioning the square into twenty rectangular regions (see fig. 15.2). If Naser's goal was to partition the square into unit squares, he should have drawn three lines vertically and three lines horizontally, which would have resulted in 16 square units. Small pencil dots, visible in each rectangle, make up the missing triangular region. Naser apparently counted the rectangles in the missing triangle to arrive at the value of 15. When Naser counted the rectangles in the missing triangle, he may have treated the segments of whole units along the hypotenuse of the triangle as whole units. This scenario would account for an area of 15 cm^2, rather than 10 cm^2, on the basis of a right triangle with a base of 4 and a height of 5.



Teresa was working on an art project. Shown below are a square and a triangle she cut out.

8 cm
8 cm
Area is 64 square centimeters
8 cm
8 cm
Area is 32 square centimeters

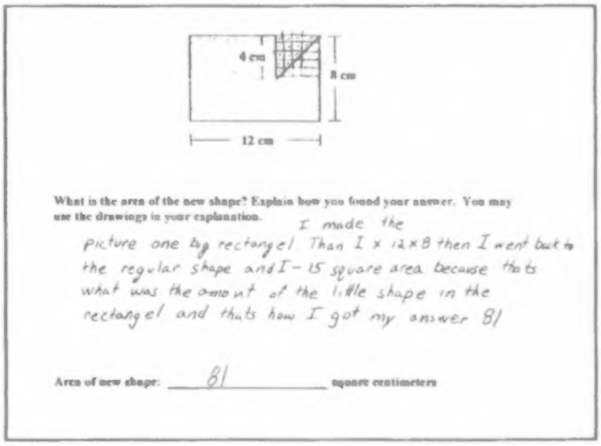
She glued part of the triangle on top of part of the square. The new shape is shown below.

4 cm
8 cm
12 cm

What is the area of the new shape? Explain how you found your answer. You may use the drawings in your explanation.

Area of new shape: ____________ square centimeters

Fig. 15.1. Irregular area task



4 cm
8 cm
12 cm

What is the area of the new shape? Explain how you found your answer. You may use the drawings in your explanation. I made the picture one big rectangel Than I x 12 x 8 then I went back to the regular shape and I − 15 square area because thats what was the amont of the little shape in the rectangel and thats how I got my answer 81

Area of new shape: 81 square centimeters

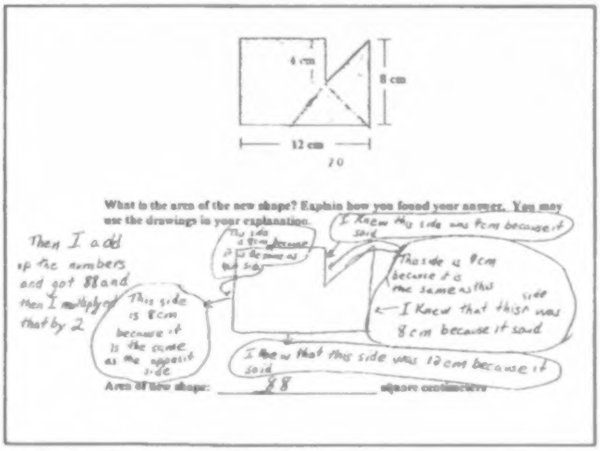
Fig. 15.2. Naser's response

Harding expressed disappointment in Naser's response. She was surprised that Naser was able to develop such an "elegant" solution yet not able to find the area of the missing triangle. When asked what made Naser's response "elegant," Harding explained, "He knew to create a larger rectangle. He knew to get rid of the little piece [referring to the triangle] . . . it was pretty good [referring to the solution]." Harding further explained that during instruction, they had not discussed how to find an unknown area by breaking the unknown region into parts with known areas. Naser's response demonstrated that he was able to extend the concepts beyond classroom instruction.

Harding also expressed frustration with a portion of Naser's response. Naser had counted the units along the hypotenuse of the right triangle as whole units. The concept of combining half units to create whole units had been directly addressed in class. Several other students had also made this mistake. Harding believed that the recurrence of this error suggested that her efforts to clarify the difference between half and whole units had been ineffective for many students.

Kevin's response

Kevin reproduced the irregular figure and labeled the length of each side (see fig. 15.3). The labeling on the text bubbles of Kevin's explanation suggests that he was aware that the sides of a square are congruent. Kevin's response also contains errors. For example, he argued that the hypotenuse of the removed triangle was 4 cm because it was the same as the leg of the missing triangle. He then added the sides of the irregular figure, and because of his incorrect work, obtained a sum of 44 as the result. Kevin found the perimeter of the irregular figure rather than the area. He also explained, "and then I multiply that by 2." This incorrect step results in his final answer of 88. Kevin may have remembered from class that calculating the area involved multiplication. By multiplying by 2, Kevin forced multiplication to appear in his work. Although his final answer to the problem was correct (i.e., 88 cm^2), he acquired this answer through a series of errors.



What is the area of the new shape? Explain how you found your answer. You may use the drawings in your explanation.

I Knew this side was 8cm because it said

This side is 8cm because it is the same as this side

Then I add up the numbers and got 88 and then I multplyed that by 2

This side is 8cm because it is the same as the opposit side

This side is 8cm because it is the same as this side

I Knew that this side was 8cm because it said

I Knew that this side was 12cm because it said

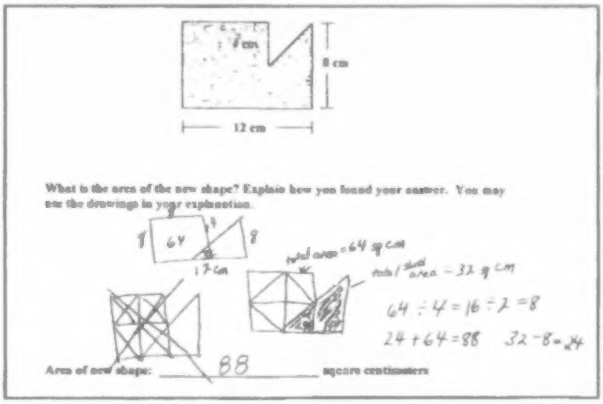
Area of new shape: 88 square centimeters

Fig. 15.3. Kevin's response

At first, Harding was pleased with Kevin's response. She said, "Kevin does not always understand math, but this time he got it." She also said that she was impressed by the detail that he provided in his response. On closer inspection, however, Harding realized that her initial evaluation was inappropriate. She had seen the correct answer and the "detail that filled the page" but had not examined the contents of the response. After reading through Kevin's response, she concluded, "He thinks he is finding a perimeter." Harding also expressed concern about Kevin's efforts to multiply by 2. She did not understand why Kevin would include this step and intended to ask him to explain his work orally the next day. If Kevin had given only his final answer, his teacher may have assumed that he had a firm understanding of the concepts being assessed.

Ning's response

As shown in figure 15.4, Ning created a series of diagrams to explain her response. She first divided the irregular figure into the overlapping square and triangle. She labeled each subdivision with the given areas. She then divided the square into four smaller squares with equal areas. She divided each resulting square in half to form two triangles with equal areas. The process that Ning used is clarified through the accompanying calculations. Using symbols, Ning divided the area of the large square (64) by 4, resulting in a value of 16. She then divided the area of the small squares (16) by 2, giving her the area of the small triangles (8). The supporting calculations clarify her pictorial representation.



What is the area of the new shape? Explain how you found your answer. You may use the drawings in your explanation.

64 ÷ 4 = 16 ÷ 2 = 8

24 + 64 = 88 32 − 8 = 24

Area of new shape: 88 square centimeters

Fig. 15.4. Ning's response

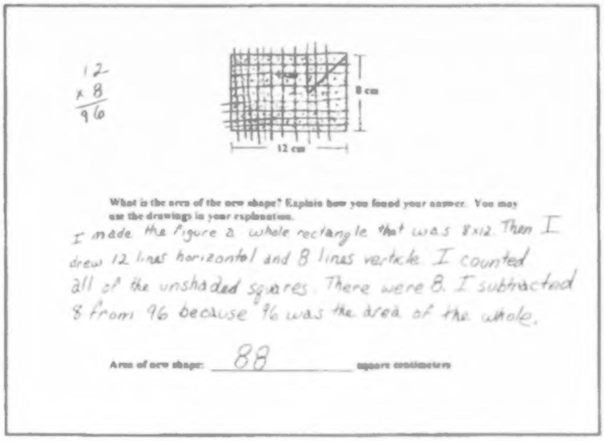
Ning's next step was to remove the area of overlap (8) between the given square and triangle from the area of the given triangle (32). Apparently, Ning recognized that this amount is accounted for in the areas of both the large square and the large triangle. Finally, she summed the resulting value with the area of the large square, obtaining the correct answer of 88.

Ning's teacher reacted to her response with great surprise. She said, "Ning's English is so flawed; I really don't have a grasp of what Ning knows and doesn't know." Ning's family had recently moved to the United States, and her English skills were not well developed. From Ning's response, Harding decided, "I need to provide her with more opportunities to explain herself through pictures. It is clear to me now . . . this might be a way to get beyond her language problems." Although Ning's response contained few words, it conveyed a great deal of understanding. Through the use of diagrams and mathematical symbols, Ning was able to effectively communicate her reasoning process.

Becky's response

Becky also produced a correct solution in response to the Irregular Area Task. Her explanation is shown in figure 15.5. Although she wrote that she had drawn twelve horizontal lines and eight vertical lines on the diagram of the irregular figure, Becky had drawn eleven vertical lines that partitioned the region into twelve vertical sections and seven horizontal lines that partitioned the region into eight horizontal sections. Although Becky incorrectly conveyed her approach in her written response, the diagram that she provided clarified the process she had used.

Becky explained that she had counted eight unshaded squares. Then she subtracted these eight from 96, the area of the whole figure. On the left side of Becky's paper, she wrote the calculation 12 × 8 and the resulting answer of 96. Becky's explanation states that she counted the units that made up the missing triangle. Examination of her diagram suggests that she also counted the number of units in the larger rectangle. In Becky's drawing, the large rectangle is subdivided into 96 unit squares. Each of these unit squares contains a pencil dot. It seems that Becky either multiplied 12 × 8 to verify the number of unit squares that she counted or counted the number of unit squares to verify her multiplication.



12
× 8
96

What is the area of the new shape? Explain how you found your answer. You may use the drawings in your explanation.

I made the figure a whole rectangle that was 8x12. Then I drew 12 lines horizontel and 8 lines verticle. I counted all of the unshaded squares. There were 8. I subtracted 8 from 96 because 96 was the area of the whole.

Area of new shape: 88 square centimeters

Fig.15.5. Becky's response

To explain how she acquired the area of the missing triangle (8), she wrote that she had counted the unshaded squares. This explanation does not appear to be supported by her diagram. In the diagram, the missing triangle is half of a 3 × 4 rectangle that has an area of 12. Half of this area results in an area of 6 for the missing triangle. However, if all the segments in the 3 × 4 rectangle are counted, including those along the hypotenuse, then the total is 16 segments. Half of 16 is 8. Becky's determination that the area of the missing triangle is 8 may have been the result of a misconception concerning how to deal with the units that are divided along the hypotenuse of the triangle. Becky also had a partially erased calculation of "96 − 16 = 80" written on her paper. Examining the answer space suggests that prior to an erasure, Becky had written "80" in the answer space. It is possible that Becky had originally sought to subtract what she believed to be the area of the 3 × 4 rectangle.

Harding thought that Becky had effectively conveyed a "basic" understanding of the area concept. Harding said, "I should be happy with this. But Ning's response was so good . . . Becky is still counting." She explained that many of her other students were also "still counting." She expressed concern that her students would choose to count 96 square units rather than multiply the base by the height. In the future, Harding intended to create a task about a figure with a large enough area that counting would not be possible. She hoped that this type of activity would convince her students that benefits were to be found using multiplication in determining an area. Harding did not identify Becky's potential misunderstanding of how to account for the units along the hypotenuse of a right triangle when finding the area.

Summary

As these examples show, students as young as sixth graders are capable of providing detailed written explanations that reflect their mathematical reasoning. A teacher should not, however, expect this type of detail the first time that students address open-ended tasks. These examples were collected at the end of sixth grade after Harding had emphasized clarity in writing throughout the year. When her students submitted incomplete or unclear written responses, Harding provided written or oral feedback that indicated which portion of the response required further elaboration.

Examining the explanations gave Harding evidence of the diverse levels of understanding among her students. Kevin's solution contained a series of errors, including an attempt to find the perimeter of the irregular figure. On the basis of a superficial observation of a correct answer and an explanation that "filled the page," Harding originally concluded that Kevin had a complete understanding of the area concept. After closer inspection, she realized that Kevin was attempting to find the perimeter rather than the area of the irregular figure. When Kevin found an incorrect value for the perimeter, he multiplied it by 2. This example raises an important concern when examining student responses. When students are asked to provide explanations, teachers should take time to read and make sense of their work.

Ning, Becky, and Naser all used reasonable approaches to acquire their answers. Ning used multiplication and division to find the area of the irregular figure. Becky used two approaches. In one approach, she attempted to decompose the irregular figure into unit squares in order to count the unit squares. In the other approach, she used multiplication to find the area of the larger rectangle and subtracted what she thought was the area of the missing triangle from this value. Harding did not acknowledge the approach that used multiplication; instead, she expressed disappointment that Becky was still counting. Naser used multiplication to find the area of the large rectangle and a counting strategy to find the area of the missing triangle. Harding was satisfied with Naser's overall approach but was disappointed that he was unable to find the correct area of the missing triangle. All three of these students exhibited flaws in their communications, but their overall explanations offered clear indications of their reasoning processes.

Using text, diagrams, and symbols in their responses to this task supported the students' communications. For example, Ning had recently moved to the United States, and her English skills were not well developed. Through the use of symbols, diagrams, and some text, Ning was able to provide a convincing argument that supported her correct answer. Text and diagrams are also likely to be appropriate methods for younger students to use to communicate their knowledge. In elementary and middle school, students are still developing their writing skills. The use of diagrams and symbols offers these students additional tools for expressing their knowledge.

Cohen and Fowler (1998) have argued that assessments should elicit evidence not only of what students can do but also of what they understand. The detailed explanations offered by her students allowed Harding to evaluate their understanding of the area concept. Time, practice, and feedback had given Harding's students the opportunity to develop their written communication skills. By examining their explanations, Harding saw their varying levels of understanding. Naser, who gave an incorrect answer, displayed greater knowledge of the area concept than did Kevin, who gave the correct answer. The freedom to use text, diagrams, and symbols in response to an open-ended task supported these students as they displayed their mathematical knowledge.

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ACTIVITIES

The Teaching, Learning, and Assessment Principles

Smith: "Redefining Success in Mathematics Teaching and Learning"

Respond to the two following reflections, which are included in the article. Be specific in your comments.

1. What do you take as evidence of your success as a teacher of mathematics? What does it mean for students to be successful in your mathematics classroom?
2. What can you conclude from the spring lessons about Henderson's new expectations for her students? How was she supporting students to enable them to meet her new expectations?

Kazemi: "Discourse That Promotes Understanding"

Explain the difference in expectations for student performance between asking students to give statements that justify procedures and asking students to just describe those procedures. Include any personal experiences that you have had either as a teacher or as a student participant in each kind of discourse.

Warrington and Kamii: "Multiplication with Fractions: A Piagetian, Constructivist Approach"

The authors describe the different solution strategies used by three students to solve the following problem:

A half of a pound of jelly beans was in a container. Georgia and Emma ate 2/3 of the jelly beans in the container. How much did they eat?

1. Describe how each of these three students would have applied their strategies to answer this problem:

   1/3 of a chocolate cake is left. John eats 3/4 of this amount. What fraction of the original cake did he eat?

2. Which strategy makes the most sense to you? Explain your answer.

Cohen and Fowler: "Create Assessments That Do It All"

1. Use the list of nine strategies given at the end of the article to create a set of five or more test questions on a particular mathematics topic.
2. Discuss how your test assesses the range of skills, knowledge, and conceptual understandings described in the article.

Reinhart: "Never Say Anything a Kid Can Say"

Identify in the article three examples that illustrate, respectively, the Teaching, the Learning, and the Assessment Principle. Explain how each example illustrates that particular principle.

3.

THE TECHNOLOGY PRINCIPLE

Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.

The effective use of technology should—

- allow students to extend their range of accessible computational problems and provide them with more examples or representational forms than are feasible by hand, thus enabling students to execute routine procedures quickly and accurately and freeing more time for making and exploring conjectures, conceptualizing, and modeling;
- engage and encourage students' ownership of abstract mathematical ideas;
- offer teachers options for adapting instruction to special student needs;
- provide unique learning opportunities in such areas as graphing, visualizing, computing, and access to the Internet and the World Wide Web;
- provide students access to powerful mathematical ideas at all grade levels by managing complex calculations, organizing and displaying large data sets, graphing functional relationships, producing models and simulations, and creating interactive graphical and geometric environments. (NCTM 2000, pp. 24–27).

Calculators and computers "are essential tools for teaching, learning, and doing mathematics. . . . They can support investigation by students in every area of mathematics, including geometry, statistics, algebra, measurement, and number. When technical tools are available, students can focus on decision making, reflection, reasoning, and problem solving" (NCTM 2000, p. 24).

The articles in this section illustrate this power of electronic technology to enhance learning. Examples are included of how computers enable students to interact more deeply with powerful mathematical ideas. Instead of focusing only on the mechanical aspects of graphing, data displays, or geometric constructions, computer-assisted instruction helps students make sense of such displays and engage with the underlying mathematics in personally meaningful ways.

Burrill addresses common myths about calculator use and the place of basic-facts instruction in the mathematics classroom. Computational skill is important, but it must be accompanied by conceptual understanding. Calculators can be powerful tools to help students gain this understanding and develop mathematical ability.

The article by Feicht illustrates how computer use can enhance students' understanding of data displays. The author requires his seventh-grade students to create for their computer-generated graphs both titles and axis labels that go beyond simple descriptors, providing evidence of each student's understanding of the data displayed. This activity helps students analyze and interpret the information created by the computer and succinctly state, in each title, the conclusions that they draw from their graphs.

McClain, McGatha, and Hodge describe eighth-grade students' work with computer-aided exploration of statistical data. The authors found the computer to be an essential tool that allowed the students to organize, describe, and compare large data sets. Giving students the options for using a variety of graphical representations to structure their data facilitated rich classroom discussions of the meanings of such displays. Placing the classroom emphasis on the quality of the explanations helped students develop strong mathematical arguments and understand the need for valid forms of justification.

The final article describes how the interactive software program The Geometer's Sketchpad can be used to develop geometric understanding at the middle school level. Students begin with a period of free exploration that enables them to become familiar with the basic capabilities of the software. At the next level of engagement, students work through a sequence of semistructured activities that uncover particular geometric relationships or develop certain theorems. At the final level, students engage in independent explorations and problem solving. These last activities require students to develop an intuitive understanding of the problem situation, form solution strategies, and pose additional questions for investigation. The authors stress that for meaningful learning to occur, students need extended exposure to, and hands-on engagement with, the software within the contexts of genuine problem-solving experiences.

Computation, Calculators, and the "Basics"

by Gail Burrill

What do you believe about calculators, computation, and the relation between the two? Just for fun, take the following true/false test.

1. If students use a calculator, they will not learn the basic facts.
2. If students have weak computational skills, they cannot do well in algebra.
3. Students today know less than ever about the basics.
4. Students should not be allowed to use a calculator until grade 8.
5. It is more important for students to understand computation than to be able to do it.
6. Students do not need drill and practice on computation.
7. Even if students can solve problems using algebra and geometry and in real contexts, it is still important to know if they really understand the basic facts.
8. If students don't do well with basic facts, you should reteach them until they learn.

So, what are the right answers? Well, much depends on your perspective, knowledge of research, and beliefs. It is important to think about some issues in perspective. Granted, many people know a cashier who cannot make change or perhaps an adult who still has to write down six sets of 25 to understand it is 150. But many people lacked these same skills 30 years ago—and in fact, fewer people then had the skills we think so vital today—but many did not need them for the lives they lived and the jobs they had.

Some facts: In 1986, a meta-analysis (Hembree and Dessart) on the effect of calculator use on basic facts indicated that, with the exception of long division, students who used calculators had the same proficiency on computational problems as those who did not. Those who used calculators scored much better on problem solving and conceptual tasks. According to the longitudinal National Assessment of Educational Progress (NAEP) results, students at grades 4 and 8 score significantly better today on computation than they did in 1973. The Third International Mathematics and Science Study (TIMSS) indicates our students score better in computation than they do in other topics. Countries that did well overall do not emphasize computational skills to the exclusion of other mathematics. In fact, curricula in these countries has a heavy focus on algebra and geometry in the middle grades.

Some thoughts: There is a certain amount of practice associated with everything we do; practice is necessary to make a process comfortable and automatic. Students do have to practice to become proficient at computation, to understand patterns that make computation easier, and to know which strategy to use when. But although practice is essential, its effect is limited without a conceptual foundation. There also comes a time to move on. I can remember trying to reteach basic facts in general math. It was almost like putting a coin into a soda machine and having the machine reject it; the coin just falls through. No matter how many times you put the coin back into the machine, it still falls through. Teaching computation over and over again is much the same; students just pass through, and none of the ideas stick.

At the time this article was written, Gail Burrill was the president of the National Council of Teachers of Mathematics. She is now on the faculty of Michigan State University. As an instructor for Teachers Teaching with Technology, she conducts workshops on using technology in the classroom. She served as the director of the Mathematical Sciences Education Board and was formerly a secondary school teacher and department chair in Wisconsin. Her honors include the Presidential Award for Excellence in Teaching Mathematics and the Wisconsin Distinguished Educator Award. She is the author of numerous books and articles on statistics and mathematics education.

To allow only students who have "mastered" computation to use a calculator seems like an inequity of the worst kind. To do so denies students access to a tool that may help them succeed in other important mathematics. Calculators are a normal part of almost everyone's world—why should school be different?

It is important for students to understand computation—I want students who recognize what magnitude their answers should be, who know when to use multiplication and not division or addition, and who understand when a decimal is a better choice for expressing a number than a fraction. If students need a calculator to carry out some of the operations but understand what they should get, they can be successful at solving problems, including those in algebra and geometry.

Some questions: Computation is critical, but how much? What set of skills and facts is essential? And what is the role of the calculator? Too often calculators are reserved for checking answers; it is time to think hard about how calculators can help students better understand mathematics. Using a calculator to answer the question "multiply" will not help students understand; students must be asked questions that force them to think about what they are doing. Calculators can be wonderful tools but should be used to help students develop greater mathematics ability—not less.

True or false: Calculators are used in meaningful ways in *your* classroom.

Making Charts: Do Your Students Really Understand the Data?

by Louis Feicht

A look at the charts in the local or national newspapers reveals that they are much flashier than the charts is typically created in our mathematics or science classes. They tend to draw conclusions for the readers in their captions and titles. While looking for ways to make students' science-fair projects stand out to the public and the judges, I thought that this use of more descriptive titles might grab the judges' attention and give the students an edge over the competition. Using conclusions as chart titles and captions became one of the most important teaching and learning ideas I have used.

I showed the class some examples from *Newsweek* and *USA Today* and asked them to start titling their charts more like professional journalists. In essence, I asked them to draw conclusions on the basis of the data in front of them and to write these conclusions in complete sentences. I realize that asking students to draw conclusions from data is not a groundbreaking idea. This request may seem easy enough, especially since the students created many of their charts from their science experiments that used real data. But the sad fact was that the overwhelming majority of the students had already formed a conclusion before even looking at the charts. And even though they had formed conclusions that may have been supported by the chart, they could not think of creative titles. They could not put their conclusions into a concise sentence.

Seventh-grade students could label two columns of data, such as "Lunch" and "Number of Students," and automatically attach the corresponding labels to the spreadsheet chart. And almost all could label the chart "Lunch versus Number of Students" (fig. 1). Although these students were very bright and had seen and made lots of charts and graphs in science and mathematics classes, they had difficulty making inferences from the chart and data. I had obviously mistaken some things about students' understanding.

A chart labeled as in figure 17.1 is now only the starting point for my students. I no longer accept these kinds of terse descriptions of data, which do not reveal students' understanding. I now require detailed descriptions of both axes and the chart title. Students may at first seem reluctant to use more detail in labeling their charts, as indicated by the following dialogue, and may have some difficulty on their first charts. But once this standard is established, the students naturally begin to analyze the data.

Lunch	Students
Mashed Potatoes and Gravy	29
Corn Dogs	10
Turkey Tetrazzini	40
Pizza	37
Tomato Soup and Grilled Cheese	5

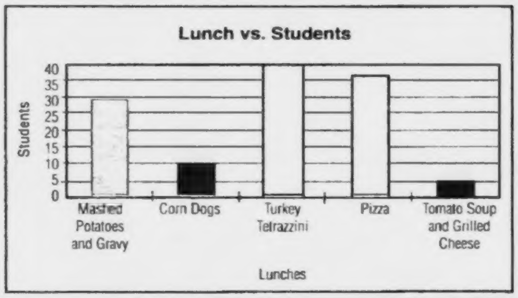

Fig. 17.1. Previously I would consider this student chart as a finished product; now I consider it only the starting point.

Before I ask students to create the titles of the charts, we first discuss the labels of the axes. I found that requiring students to formulate descriptive labels for

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the axes is an important prerequisite for their understanding of the data. These discussions lead to colorful, attention-grabbing titles. A typical conversation for labeling axes and titles follows:

The Horizontal Axis

T: What does the horizontal axis represent?

S: Lunches.

T: What does it tell us about lunches?

S: I don't know. It tells us about lunches at school.

T: At our school?

S: Yes.

T: Do we serve this kind of food at our school?

S: Yes.

T: Imagine I didn't know anything about the food at this school. I bring my own lunch everyday. How could you give me more information about the horizontal axis?

S: I don't know.

T: Well, suppose I look at the chart and I see the label "Lunches." Does it mean the lunches served at all schools in the world, the lunches served in the state of Oregon, the lunches served at the state penitentiary in the month of August . . . .

S: No, it means the lunches we serve at Silver Crest School.

T: Yes, very good. Could you repeat that for me one more time?

S: The lunches we serve at Silver Crest School.

T: I like it; use it for the label of the horizontal axis.

The Vertical Axis

T: Now let's do the vertical axis. What does it tell us about?

S: Students.

T: Suppose I go over to the category for turkey tetrazzini on the horizontal axis. And go up, what does the chart tell me?

S: Forty students rated turkey tetrazzini as their top choice.

T: How many students?

S: Forty.

T: How many students rated mashed potatoes and gravy as their top choice?

S: Twenty-nine.

T: How many students rated corn dogs as their top choice?

S: Ten.

T: So what information is the vertical axis giving us?

S: The number of students.

T: The number of students that . . . what?

S: The number of students that like a certain type of food.

T: Yes. But not just any food, it can't be a cup of coffee and piece of pie, can it?

S: No. [Exasperated sigh!] The number of students that picked a school lunch as their favorite.

T: Sounds good to me; use it for the title on the vertical axis.

The Title of the Chart

T: Okay. Now we need a title for the chart.

S: Lunches versus students.

T: No. I want something more descriptive. Can you look at the chart and form a conclusion?

S: Students versus lunches?

T: No. I would like a complete sentence. The title must tell me more about the chart. Suppose the superintendent is going to use this chart to change the lunches offered to students next year on the basis of popularity of the lunch. What would he think from looking at the chart?

S: That turkey tetrazzini was the favorite.

T: The favorite of whom, the cooks?

S: Noooo [another exasperated sigh!], the students.

T: Okay. You've drawn a conclusion. Could you put it into one complete sentence?

S: Turkey tetrazzini was the students' favorite lunch.

T: Let's get even more precise. Did all the students choose turkey tetrazzini as their favorite lunch?

S: Forty students did.

T: Can we work that into the title?

S: Forty students chose turkey tetrazzini as their favorite lunch.

T: Yes, now we are getting much closer. Is forty students more than for any of the other lunches?

S: Yes, how about "More Students Prefer Turkey Tetrazzini Than Any Other Lunch"?

T: Excellent. I like that; put it on the chart, and let's see how it works.

[The student enters the title into the spreadsheet chart.]

T: That looks good. I think we could get a little more precise. Which students are we talking about?

S: Silver Crest students.

T: Yes, let's add that. And what type of lunches are we talking about—home lunches? School lunches?

S: School lunches.

T: Okay. Let's put that in, and I think we have a winner!

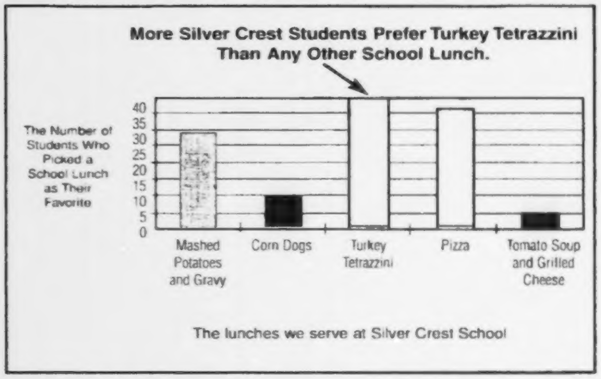

Fig.17.2. A chart that uses sentences to label axes and title. The title indicates a student's conclusions inferred from the data.

Figure 17.2 shows the new version of this student's chart. I found that having students talk about and then put into writing their labels and titles helps them develop their thoughts and interpret what they see. Request that students do these steps whenever possible, and you will soon see some amazing results. I have often had students who observe information in data that was not immediately obvious to me. You will get incredible, creative titles, and students will begin to use several subtitles or captions to label their charts, as can be seen in figure 17.3.

By requiring students to analyze and draw accurate conclusions on the basis of data, teachers can help students develop a valuable analytical skill.

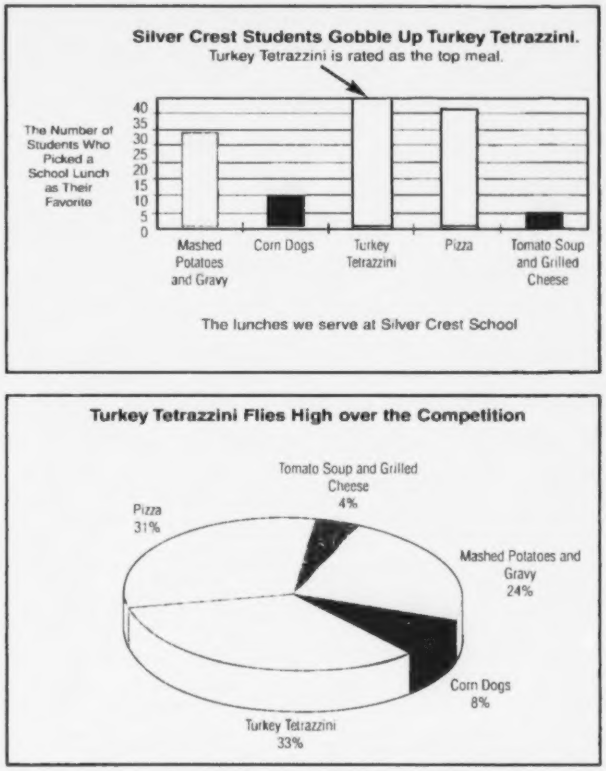

Fig.17.3. Two typical student charts after students gain some experience making informative titles.

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Improving Data Analysis through Discourse

by Kay McClain, Maggie McGatha, and Lynn L. Hodge

The National Council of Teachers of Mathematics has been advocating the importance of effective communication in classrooms since the release of its *Standards* documents (NCTM 1989, 1991). This emphasis is echoed in Richards's (1991) description of an inquiry classroom (see also, e.g., Ball [1993]; Cobb, Wood, and Yackel [1991]; Lampert [1990]). In this setting, the teacher's role is to guide the negotiation of classroom norms to enable the teacher and students together to engage in meaningful mathematical discussions, which include asking questions, solving problems, posing conjectures, and formulating and critiquing mathematical arguments. An increased emphasis on communication in the mathematics classroom allows students the opportunity to discuss and validate mathematical ideas and to make and evaluate conjectures and arguments.

We find it particularly important to focus on communication when planning to teach statistical data analysis. In recent years, statistics has received increased attention in many education reform documents (cf., NCTM [1989, 1991]; Shaughnessy [1992]). Such discussions center on the increasingly prominent role of statistical reasoning in work, community, and home activities. The task for students, then, is to participate in both developing and critiquing data-based arguments.

This article describes a classroom in which students developed statistical understanding of exploratory data analysis through mathematical argument. The session is taken from an eighth-grade classroom in which we worked with students on a data-analysis unit. The students had also participated with us in a data-analysis unit during the fall semester of the previous year. Our overall goal was to develop two instructional sequences, one for seventh grade and one for eighth grade, that focused on proactively supporting middle school students' development of statistical reasoning. As part of our efforts, we viewed the development of mathematical argument as a fundamental aspect of the classroom environment. As a result, the goal for the students' development was twofold—to support their ability to reason statistically as they developed ways to create and justify analyses of data.

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Instructional Sequence

As we designed the instructional sequences, we tried to identify the "big ideas" in statistics. We hoped that these themes could guide the design of the sequence and allow us to avoid creating a list of separate, loosely related topics that characterize middle school curricula. In identifying the big ideas, we focused on the notion of distribution. This emphasis let us treat such concepts as mean, mode, and median and other ideas, including skewness and spread, as characteristics of distributions. Our focus also allowed us to view various conventional graphs, such as histograms and box-and-whisker plots, as different ways of structuring distributions. The instructional goal was to support students' ability to view data as distributed along a space of possible values instead of as a set of individual data points.

In developing the instructional sequences, we were also guided by the idea that computer tools would be essential in supporting our goals. Such tools would let students organize, describe, and compare large sets of data. We also acknowledged the importance of students' understanding how to reason with graphical representations, such as histograms and box-and-whisker plots, without concentrating instruction on the procedures for constructing these graphs. The computer tools that we designed offered a range of ways to structure data that supported students' ability to reason statistically while building toward conventional graphs.

The classroom session described here used a computer tool that displays data along a horizontal axis as in an axis plot. The tool can display two data sets simultaneously, such as those in figure 18.1, allowing for a comparative analysis. The tool also includes a number of options for structuring and analyzing data, some of which are precursors to conventional methods. For example, students in our classroom session used two options: (1) partitioning the data into four equal groups so that each group contains one-fourth of the data, as a precursor to creating the box-and-whisker plot; and (2) partitioning data into two equal groups so that each group contains one-half the data. These two options, in addition to the role of the computer tool, are described in greater detail later in the discussion of the class session.

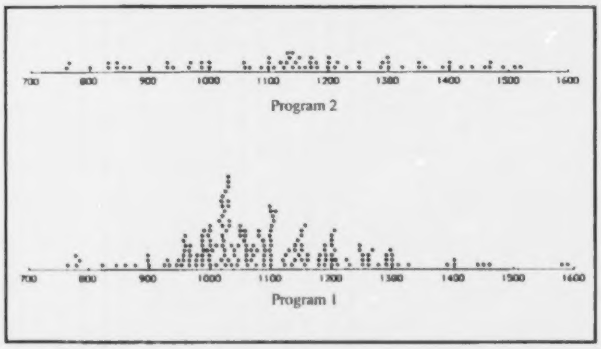

Fig.18.1. SAT preparation program data

In planning the instructional sequences, we reasoned that students needed to develop arguments based on the reasons for generating the data. They also needed to develop ways to analyze and summarize the data to substantiate their arguments. We anticipated that these goals could be achieved by developing instructional tasks that involved describing a data set or analyzing two or more data sets to make a decision.

Classroom Session

The classroom routine for our instructional sequences began with introducing a task by talking through the data-creation process. If students did not collect the data themselves, it was still important for them to discuss the reasoning behind the data-collection activity. Students would typically make conjectures and offer suggestions about the information necessary to make informed decisions. They would discuss what data they needed and how they might collect those data. These discussions were essential in grounding the students' recommendations or arguments in the context of the data-analysis activity. After this discussion, the students worked in pairs at the computer to conduct their analyses, writing justifications for recommendations and developing graphs of the data to support their recommendations. These justifications formed the basis for subsequent whole-class discussions. A projection system allowed students to display their data sets as they discussed their ways of organizing the data to support their decisions.

Whole-class discussions often involved critiquing these data-based arguments to emphasize the importance of designing adequate reports that were justified in terms of the task situation. Many discussions centered on developing adequate arguments and graphs to substantiate students' recommendations. These types of discussions were initially difficult for the students because of their lack of experience in engaging in mathematical argument. They had trouble distancing themselves from the classroom and understanding what someone outside their own investigations would need to know. As a result, their explanations often lacked detail; however, as students participated in such discussions more often, they began to support and articulate their arguments more effectively.

One of the task situations posed during the eighth-grade sequence involved members of the parent-teacher organization (PTO) in gathering data about two SAT preparation courses so as to recommend the more beneficial course. Students shared information about such programs and discussed why they might enroll. They then generated a list of information needed to make an informed decision about a program's success and discussed how to gather these data. Students were then asked to analyze data from two programs in which the SAT scores of the participants at the completion of the programs were shown on axis plots (see fig. 18.1). Students were asked to use their analyses to provide information that the PTO could use in formulating its recommendation. In their preliminary discussions, students reasoned that the "samples" taken from the two courses should have students of equal ability for the task to be reasonable; they operated under that assumption as they conducted their analyses and developed their arguments.

After the initial discussions, student pairs conducted their analyses using the computers. At the end of class, they were asked to turn in reports based on their analyses. The teacher reviewed these reports to decide how to orchestrate the whole-class discussion for the following day. To help support students' mathematical development and to assist them in formulating coherent arguments, the teacher began with one report that had the potential to foster a discussion about the adequacy of reports and graphs. The report selected was based on the notion of consistency, arguing that program 1 was better because the scores of its participants were more consistent. The report noted that most of the scores in program 1 fell in a range of about 300 points, whereas the scores in program 2 fell across a much wider range.

After the report was read in class, Barry made the following observation:

Barry: I'm not trying to put this way down, but I don't think it's a very good way to do on this particular thing because on this, we're not trying to pick out which one is more consistent—we want the highest grade possible...you'd rather have a higher chance of getting a higher grade. That's what you're looking for, not consistency.

Teacher: So you're saying, in this particular case, for this particular problem, you're not sure that consistency would be what you would want to know about.

Barry: Correct.

In earlier analyses, the notion of consistency had emerged from the students' investigations. They had reasoned that in some instances, being able to predict what might happen was more important than having a "better score." This notion first emerged in analyzing braking distances of different cars. The students reasoned that having a car that might stop quickly one time but slowly another time was less preferable than having a car that had a consistent braking distance. The task context provided a basis for their argument. In this task, however, Barry was challenging the notion of consistency as sufficient to justify a recommendation. For him, consistency was not as important as the "chance of getting a higher grade."

The teacher reproduced the graph from the report on the whiteboard (see fig. 18.2). The two students had drawn a line to show the shape of the data. Other students began to discuss the adequacy of the graph. One of the students who had worked on the report attempted to explain the graph by arguing that the data points were "under the line," and for program 1, "there's a ton right in the middle" but for program 2, "they're very spread out." He went on to add that with program 2, "you have a chance of getting really low or a chance of getting really high."

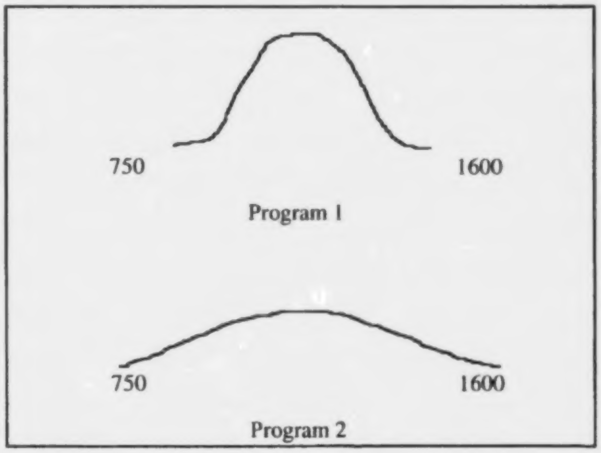

Fig.18.2. Student graph of SAT preparation program data

Next, Marcia offered her opinion about the adequacy of the report: "I think that way would work, but you have to make a lot of choices. You have to decide whether you want to go with the one that has the higher score but also the lowest score, or if you want to go with the one that is more consistent." The other students agreed that it appeared from the graph that program 2 offers a greater chance of a higher score but also a greater chance of a lower score. Program 1 offers more chance of having a score "in the middle." Because neither of these results gives information about which program is better, however, the graph and analysis offered what the students ultimately agreed was an "inadequate" way of making a recommendation.

At this point, the teacher asked Sam and Robert to share their recommendation. In monitoring their activity the previous day, the teacher saw that Sam and Robert had first looked at the data sets partitioned into four equal groups but had reasoned that the top 75 percent of each data set was in the same range, so structuring the data in that way was not useful. They then partitioned the data into two equal groups and found that the top 50 percent of program 2 was in a higher range than the top 50 percent of program 1. Sam and Robert used this justification for selecting program 2. After they explained their analysis, the teacher restated their recommendation and highlighted their justification by marking the data sets projected on the whiteboard as shown in figure 18.3.

Teacher: Their way was to do two equal groups. They found that way helpful because the upper 50 percent of this group [pointing to data on program 2] was in a range that was higher than the upper 50 percent of this group [pointing to data on program 1].

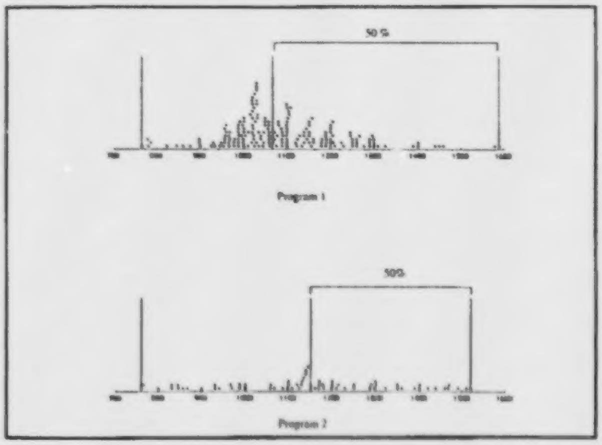

Fig. 18.3. SAT preparation program data organized in two equal groups

[Marcia raises her hand.]

Teacher: Marcia, did you want to raise an issue about their way?

Marcia: The way they had it with four equal groups, he said he didn't get anything out of the four equal groups . . . but if you look at it this way [going to the board and drawing on the diagram, as shown in fig. 18.4], you have 50 percent up here and 75 percent right here. I'm saying look at it the other way. Instead of using the 75 percent where they are the same, flip it over and do it where they are different. That's what was bothering me when he said he didn't get anything out of this.

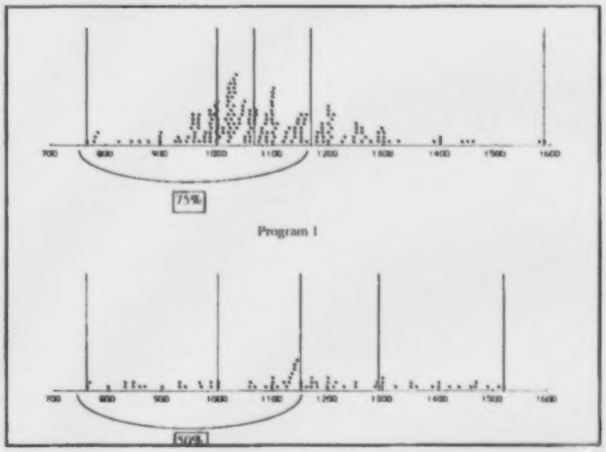

Fig. 18.4. SAT preparation program data organized in four equal groups

Marcia's reasoning used the data as they would be structured in a box-and-whisker plot. She justified why program 2 should be recommended over program 1, by reasoning about the proportions of each set of data that fell in a certain range. From Marcia's point of view, the lower 75 percent of program 1 was in the same range as only the lower 50 percent of program 2; therefore, program 2 had the better results. As Marcia spoke, the teacher drew on the whiteboard the ranges shown in figure 18.5. When Marcia finished, other students in the class agreed with her analysis. Their comments included, "Oh, wow!" "Yeah, I see!" "Oh, now I see." "I got what she is saying." Unfortunately, the bell indicating that class was over ended the discussion.

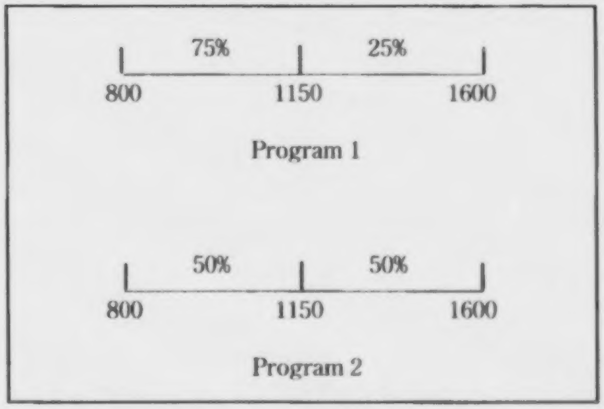

Fig.18.5. Marcia's view of the SAT preparation program data

Concluding Remarks

In this class session, the discussion focused not only on the results of the students' analyses but also on what counted as an acceptable explanation. This type of discussion supported the students' development of strong mathematical arguments. As students discussed their justifications, they also developed ways of reasoning to support their analyses. The students came to understand that merely showing a graph and making a recommendation are insufficient. The recommendation must be justified in the context of the data.

In addition, the students developed language to use in formulating arguments, as evidenced in the discussion of consistency. This significant shift required the students both to make a recommendation and to justify the recommendation on the basis of their analyses. The discussions emphasized the ways of structuring the data to support the argument. It is interesting to note that students often changed their initial judgments on the basis of the whole-class discussions—their reasoning was constantly being challenged and modified in light of others' arguments.

In discussing the expectations and obligations of this class, many students remarked that they felt as if they were "all sort of learning together." Other comments showed the importance of discussions; one student noted, "[I] liked the discussions because I can share ideas and get some back. Even when people disagree, it makes you think." In this setting, the students agreed that "everybody is smart." They commented, "Most of the time, teachers don't care about what you think, but in here we get to share our opinions—that's the whole discussion." For these students, learning mathematics was not an individual effort but depended on the contributions of everyone in the class. Their work was not complete after they had conducted their analyses. They then had to work to understand others' analyses and question the recommendations made on the basis of those analyses. Students also viewed the role of the teacher differently: "It's like the teacher teaches you, but you also teach the teacher." In this way, the students and teacher contributed to establishing classroom norms for explanations and justifications.

Engaging students in productive mathematical discussions often involves changing the students' ideas about what it means to know and do mathematics—it involves engaging them in the process of change. Instead of the teacher's assuming the primary responsibility for classroom discussions, students play an essential role in this process. This statement is not to imply that the teacher is no longer responsible for the mathematics. The teacher's role is expanded to include facilitating discussions in which students explain and justify their mathematics. This outcome is often accomplished when the teacher is aware of students' different solution methods and can orchestrate a discussion in which fundamental mathematical ideas and ways of reasoning form the basis for argument. In this way, students' communicating their mathematics becomes the basis for learning that is shared by all members of the classroom community.

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Exploring Geometry with Technology

by Azita Manouchehri,
Mary C. Enderson,
and Lyle A. Pugnucco

The study of geometry in grades 5–8 should incorporate opportunities for students to engage in exploring and analyzing geometric shapes to conjecture about geometric relationships through data collection and model construction, according to the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). In this fashion, students will develop an intuitive understanding of geometric concepts and learn to reason formally and informally. Moreover, it is hoped that through such processes, students will formulate relevant definitions and theorems. The *Standards* document also encourages the use of computer technologies in middle school mathematics instruction. This suggestion was based on the assumption that interactive environments provided by appropriate geometry software have the potential to foster students' movement from concrete experiences with mathematics to more formal levels of abstractions, nurture students' conjecturing spirit, and improve their mathematical thinking. Although the NCTM's visions for the geometry curriculum and for methods of teaching geometry in the middle levels are certainly attractive, many teachers are concerned about what software is useful for the middle school population, how such software can be used in instruction, what issues are associated with their use, and what the consequences are of learning and teaching mathematics within such environments.

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In this article we attempt to address some of these concerns by describing how one particular piece of software has allowed us to implement many of the NCTM's recommendations in our classrooms. We describe how we introduce the software to our classes, the types of activities we use with our students, and students' reaction to technology-based geometry instruction.

Introduction of the Software

One piece of interactive software that is gaining much popularity in mathematics classrooms is The Geometer's Sketchpad (Jackiw 1995), which we refer to as GSP. This software allows the construction of geometric objects and permits the measurement of many attributes. The software is dynamic and user driven and is designed to give students a workspace or worksheet and various tools that construct points, circles, lines, line segments, and rays. These construction tools are presented on a side menu of each workspace and can be used by clicking and dragging the desired tool icon (see fig. 19.1).

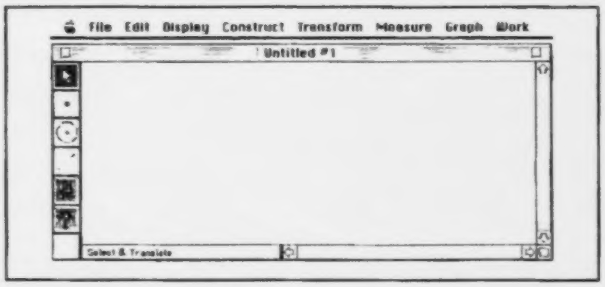


Fig. 19.1. A GSP worksheet

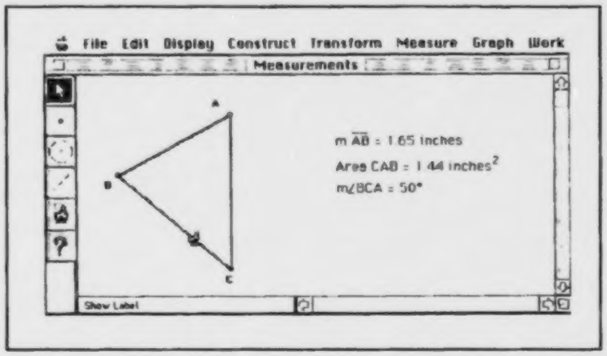
Geometric figures can be constructed with Euclidean construction methods or transformational-geometry techniques. Once constructed, a figure can be transformed by dragging one or more of its component parts by simply clicking the mouse on that part and dragging it. All constructed geometric relations are preserved during these adjustments. Measurements of segments, arc lengths, areas, perimeters, and angles can be obtained and used for calculations and explorations. To obtain such measurements, the user needs to identify appropriately the component being measured. For instance, to measure a given angle, three points defining the angle must be identified by selecting them in the proper sequence—the vertex bound between the other two points. To measure the area or perimeter of a geometric figure, all the vertices of the region must first be selected in consecutive order. This process allows the user to identify the area bounded by the vertices. Then, by choosing the "Construct" option in the menu, the user must select "Polygon Interior." This operation fills and highlights the polygonal region and gives the option of selecting the area or perimeter feature under the "Measure" menu item.

GSP software also affords opportunities to experiment with transformational geometry. The "Transform" option in the menu of the workspace allows the user to reflect, dilate, and rotate geometric figures already drawn or constructed in the workspace. To activate these options, the user has to select appropriately the figure to be transformed. For instance, if the user expects to reflect a figure about a line, the line must first be identified as a mirror of reflection, and then the object to be reflected through that mirror is selected. To rotate an object, a center of rotation must first be identified by selecting a point. Next, the object to be rotated about that point and the angle of rotation must be designated. Components of the created images, resulting from a reflection or rotation of objects, or both, change as any adjustments to the original figure occur. For example, if an angle of an object is changed by moving one of its vertices, then the corresponding angle in the image also changes in an identical fashion.

GSP software also possesses automated programming capabilities in which a particular script, or account of instructions, for a construction can be written and recorded. This procedure requires the user to complete one of two tasks: record a construction as it is constructed or copy and paste the geometric image to a text recorder, called a script. Once this recorded file is saved, it can be resurrected whenever a need arises for building and using the same model.

First Exposure: Free Exploration

As an introduction to the software, we allow students time for free exploration; they experiment with options under menu items and in the tool box. During this time, usually one to two class sessions, students are given a brief handout about the basic characteristics of the options, which include names of tools; ways to select and label items; and methods of measuring lengths, angles, areas, and perimeters of different geometric figures (see fig.19.2).



m $\overline{AB}$ = 1.65 inches
Area CAB = 1.44 inches^2
m∠BCA = 50°

Fig. 19.2. The measurements correspond with triangle *ABC*.

We have discovered that this "exploration" time helps students build the necessary basic knowledge about the capabilities of the software. This knowledge is of vital importance in later sessions when students are assigned independent explorations and problem-solving activities. In our experience with first-time users at elementary, middle, and high school levels, we noticed that during the first computer session, many important questions arise that facilitate movement from the free-exploration level to a more organized investigation. Regardless of the students' level of mathematical knowledge and background, these questions emerge as a consequence of their experimentation with GSP options. Students attempt to understand what can and cannot be done with the tools. These questions often lead to the formulation, introduction, or review of some important definitions from Euclidean geometry. For instance, as our students explore the transformation option of the menu, their encounter of the terms *dilation*, *rotation*, and *reflection* immediately stimulates them to find the meaning of the terms as well as how to use them. In such instances, we help students make connections between the mathematical logic embedded in computer constructions and the compass-and-straightedge constructions of Euclidean geometry. Our basic approach is to encourage students' use of trial-and-error strategies for such constructions and to draw logical conclusions on the basis of the result of their actions.

In this free-exploration stage, several different patterns of activity emerge among students. Students' actions are primarily guided by their own intellectual energy and curiosity. In addition, their activities are greatly influenced by the insights of their peers as they work together. Often, as one student constructs, others attempt to create similar objects. This interaction initiates conversation about how others construct figures and what they see as important issues in this process. In another situation, as one student measures the sides of the figure, classmates are enticed to mimic the same actions. Throughout the process of free exploration, we not only allow but encourage students to walk around the room and examine peer work. In our experience, this strategy has been a powerful vehicle for creating a social-learning environment in which peer talk becomes a guiding mechanism for learning.

Second Exposure: Semistructured Activities

In the second level of exposure of students to the software, we plan semistructured activities for students. These types of activities are designed with specific mathematical goals in mind and motivate students to use particular tools of the software.

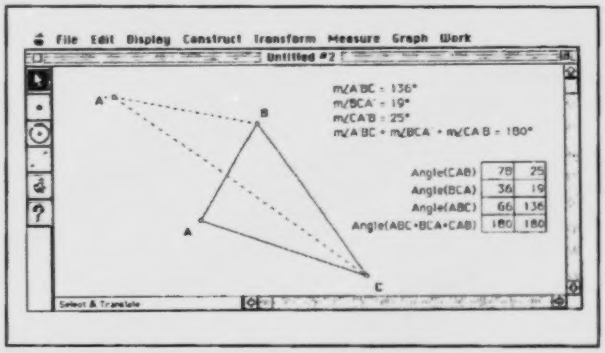
These semistructured tasks are planned to facilitate students' discovery of specific geometric relationships or their realization of certain theorems. In general, these types of tasks are separated into two categories. One type involves the use of environments that are preconstructed before class and stored on the computer. Students open the document, explore the environment, and answer questions related to data collected from geometric construction and numerical evidence. We use preconstructed environments when construction is not a primary goal of the lesson or when the construction process takes too much class time. As students open files, they can alter almost all elements of the figures already constructed. They can introduce auxiliary lines and create any additional features in the environment.

The second type of semistructured exploration requires students to construct the model that we want them to explore. We provide students with specific instructions for construction. Although we accompany these explorations with guiding questions, the students have complete control over how and in what way they choose to collect data to respond to the questions or tasks at hand. These guiding questions encourage exploration and experimentation. We usually begin by investigating attributes of triangles. This beginning evolves naturally into exploring corresponding attributes in other polygons. In this fashion, we encourage students to make conjectures and use inductive reasoning to discover and construct relationships among different classes of figures.

One such semistructured activity that we used in our classes for middle school mathematics students and for teachers follows.

Self-constructions: triangle interior-angle sum

We begin this activity by asking students to place any three points on a workspace and construct a triangle by connecting those points with line segments (see fig. 19.3). Next, students find the measure of angles *A*, *B*, and *C* in triangle *ABC* by selecting appropriate vertices and calculate the sum of all angles in the triangle with the computer's "Calculate" option. We encourage students to move any of the vertices, for example, *A* to *A'* in figure 19.3, and to look for changes in the geometry and in the numerical and visual data within the workspace.



$m\angle A'BC = 136°$
$m\angle BCA' = 19°$
$m\angle CA'B = 25°$
$m\angle A'BC + m\angle BCA' + m\angle CA'B = 180°$

Angle(CAB)	78	25
Angle(BCA)	36	19
Angle(ABC)	66	136
Angle(ABC+BCA+CAB)	180	180

Fig. 19.3. When the user moves *A* to *A'*, the new angle measurements are recorded.

As students spend time experimenting with different types of triangles, they attempt to move the vertices and position them to form a right or an equilateral triangle. At this point, many students are prepared to make conjectures about the angle sum in their triangles. This initial experimentation leads smoothly into other activities that are focused on determining the sum of interior angles of other polygons. Moreover, one recurring and intriguing inquiry that students pursue is the angle-sum property for concave polygons.

Because constructing figures and collecting measurement data are simple, students not only examine fundamental and obvious cases but explore sophisticated ideas from the middle school curriculum and beyond. We have witnessed students' enthusiasm and excitement as they generate their "own theorems" and attempt to "prove" them. As an extension of this activity, our students investigate the necessary criteria for the existence of a triangle. Since accuracy in data is essential for this particular exploration, we use a preconstructed environment (see fig. 19.4). The environment is a simple presentation of three line segments in which all are of variable length. Images of segments *AC* and *BC* are indicated by dotted lines and appear only when a triangle exists.

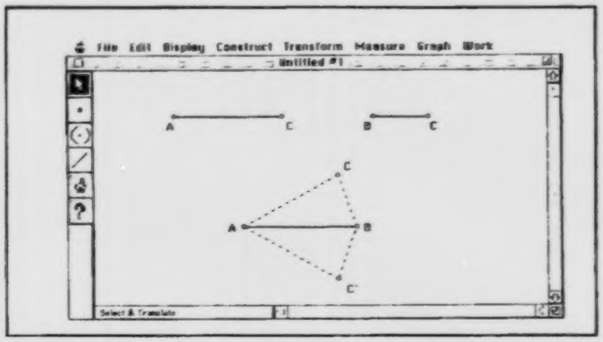


Fig. 19.4. Triangle *ABC* is built on the given lengths *AC* and *BC*.

We present the situation to students and pose questions that include the following:

- Given three specific segments, *a*, *b*, and *c*, is it possible to create in the workspace a triangle with lengths of these segments? Is it possible to create more than one triangle? Why or why not?
- Given a specific angle bounded between two segments, is it possible to create a triangle? Is it possible to create more than one triangle? Why or why not?

As students generate responses to such situations, they generally progress into exploring other related ideas that can be constructed on an individual basis.

- Can a triangle exist with any three given segments *a*, *b*, and *c*? Why or why not?
- Generate several other examples, using various lengths of segments, and attempt to make some generalizations about the existence of a triangle on the basis of its side lengths.
- Do special conditions exist for which a triangle can be formed with two given segments and one angle?

Third Exposure: Independent Explorations and Problem Solving

As students gain more experience with software through their work in semistructured and structured explorations, we introduce activities that are more open-ended in nature. In these situations, we give students a problem only; they have complete autonomy over how and in what way they use the software. This type of problem-solving activity is powerful: It affords students the opportunity to develop an intuitive understanding of the problem, form strategies for finding a solution, and pose additional questions and problems. An example of such an activity follows. This activity deals specifically with similar triangles. Although it is recommended for high school students (Coxford et al. 1991), we have had a great deal of success in using this activity with middle school students.

> Given a triangle, locate the midpoints of the sides. By connecting the adjacent midpoints, determine if a relationship exists between the area of the triangle defined by the midpoints and the original triangle (see fig. 19.5).

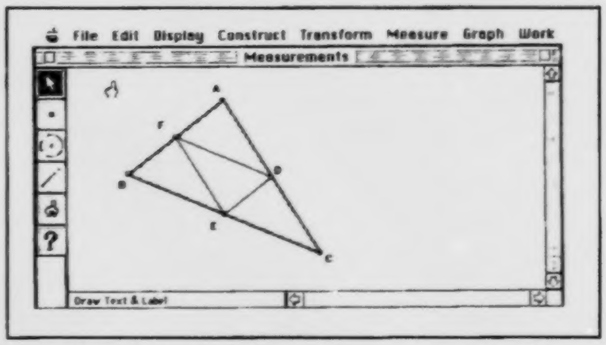


Fig. 19.5. Triangle *FED* is formed by connecting the adjacent midpoints of the sides of triangle *ABC*.

We presented our students with the task and allowed them to tackle the situation with the aid of the GSP. When students explored this problem with technology, the outcomes were significant, both in the context of their developing problem-solving strategies and in the unfolding of significant mathematical relationships.

On introduction to the problem, students immediately constructed a triangle on the workspace, identified the midpoints, and connected adjacent midpoints to form the interior triangle. After animating the original triangle for a few seconds by moving its vertices and without using the measurement tool, one student observed that the four triangles inside were of the same size and shape and that the area of the interior triangle was one-fourth the area of the largest triangle. We insisted that students formulate a convincing argument for verifying this observation. As a group, several students measured the areas of interior triangles and compared the sum of those areas with the area of the exterior triangle. This procedure was acceptable, and once completed, students assumed that the "evidence" was reasonable for their initial conjecture.

We argued that although the students had established that the areas were equal, they had not yet demonstrated that the four triangles inside were congruent. We assigned them the task of convincing an outsider that the interior triangles possessed the same shape and size. This task created a great deal of dialogue and resulted in various groups' taking several different positions. Some groups attempted to establish equality by measuring the sides and angles of each triangle and demonstrating equality through numerical data. This approach, in fact, was the initial procedure used by most students. We asked whether students could form an argument that did not build on measuring sides and angles. One group used the copy-and-paste function of the software to make a copy of one small triangle and then compared it with the other three triangles. This group argued that since they could see the exactness of triangles by matching their corresponding sides and angles, the four triangles had to be congruent. This strategy was insightful and was well received by everyone in class. Meanwhile, another group suggested performing the matching procedure by using the transformation capability of the software—rotating and reflecting one of the triangles and matching it against the other three triangles.

Once this verification process was completed by all groups and for several different triangles, students were prepared to move to a more formal line of reasoning about why the triangles were congruent. Indeed, several students decided to investigate whether repeating the process of bisecting the sides of smaller triangles would lead to a pattern. After students spent some time investigating ratios for several such cases, they reached the conclusion that to predict the ratio of the area of the smallest triangle to that of the largest, they needed only to multiply 4 by itself the same number of times as the level of iteration. The students not only generated the exponential function describing the iterated-growth pattern but also inquired whether the same relationship was true for other polygons, including quadrilaterals, pentagons, and hexagons. This extension itself served as another student investigation in the classroom.

Issues Surrounding Technology in the Classroom

As illustrated in the previous example, immediate feedback through interactive software greatly facilitated students' experimentation. Moreover, the ability to generate finitely many cases of different figures encouraged a conjecturing spirit and provided an opportunity to test those conjectures, which is often not possible with paper-and-pencil investigations.

The foregoing classroom encounters also present concrete examples of how some of the NCTM's recommendations for geometry curriculum (NCTM 1989) might be achieved. In our experiences, the facilities provided by the interactive computer software empower students to investigate concepts that were once thought unattainable both in scope and in depth by middle school students. In particular, students explore transformations of geometric figures, represent and solve problems by using geometric models, and make connections to such other mathematical areas as algebra. Technology enables teachers to create an exciting and productive learning atmosphere in which the focus of geometry is not on memorizing definitions and formulas but on investigating and using geometric concepts and relationships.

Although computers permit us to envision sweeping changes in the teaching of middle school mathematics, several issues need to be considered as teachers plan and design computer-enhanced instruction. Perhaps the most important point is that the changes suggested by the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989) do not occur as the result of one or two episodic student experiences with technology. Through repeated exposure and continued hands-on experiences with computers and problem-solving situations, the true spirit of the *Standards* document gains life in the classroom. This scenario demands that teachers also be willing to learn along with their students, to experiment with ideas, and to learn to teach with technology. Indeed, an important aspect of designing and implementing new curricular and instructional ideas is collaborating with colleagues and sharing productive ideas.

The final point is that the learning potential facilitated by interactive software does not diminish the importance of teachers' help but actively endorses teachers' contributions. Helping students can take the form of asking "why" questions, provoking "conflict" when students overgeneralize on the basis of examining only a few cases, and assisting in a search for counterexamples when they are not clearly obvious. In the process of asking "what if" and "what if not" questions, the teacher facilitates students' growth from an intuitive level to a more formal level of thinking.

Concluding Remarks

The mission of traditional geometry instruction in the middle grades has been one of training students to memorize relationships, definitions, and methods of measurement and construction, with the goal that they can ultimately generate proofs. Currently, the goal of mathematics instruction at the middle grades, as acknowledged by the National Council of Teachers of Mathematics (1989), is one of improving students' mathematical thinking and their ability to communicate thoughts clearly. This expectation implies going beyond a superficial introduction of basic concepts by moving toward more in-depth investigations. To fulfill these goals, any innovative geometry curriculum for the middle grades should capitalize on building students' geometric intuition by using a variety of useful tools for their experimentation. Appropriate computer software supplies the teacher with a powerful instructional tool. This tool can facilitate the creation of a rich mathematical environment that students can explore. It fosters a conjecturing spirit that is fundamental for advancement in mathematics and lifelong learning.

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ACTIVITIES

The Technology Principle

Burrill: "Computation, Calculators, and the 'Basics' "

The author poses the three questions listed below. Write a reaction paper giving your personal response to these questions.

- Computation is critical, but how much?
- What set of skills and facts is essential?
- What is the role of calculators in the classroom?

McClain, McGatha, and Hodge: "Improving Data Analysis through Discourse"

Explain how the activities described in the article do or do not exemplify the aspects of the Technology Principle listed in the introduction to this section. Be specific.

Part 2:

Content Standards

THE TEN Content and Process Standards "describe a connected body of mathematical understandings and competencies — a comprehensive foundation recommended for all students, rather than a menu from which to make curricular choices" (NCTM 2000, p. 29) Although, the Standards apply across all grades, emphasis on particular content and processes will vary with grade level. Growth along these Standards is expected across grades, with students achieving a certain depth of understanding at prescribed points in the curriculum.

In this book, the division of the ten Standards into two sections, parts 2 and 3, is an organizational decision. Within the complex reality of the classroom, the Content and Process Standards are intimately intertwined. "Because mathematics as a discipline is highly interconnected, the areas described by the Standards overlap and are integrated. Processes can be learned within the Content Standards, and content can be learned within the Process Standards. Rich connections and intersections abound" (NCTM 2000, pp. 30–31).

The five Content Standards covered in part 2 describe expectations for the understanding, knowledge, and skills that students should develop through encounters with sound, high-quality mathematics at all levels of the curriculum. The articles illustrating these Content Standards are organized into three groups on the basis of the related nature of particular areas. Section 1 covers both Number and Operations, and Algebra. Section 2 includes the two Standards on Geometry and Measurement, and section 3 addresses Data Analysis and Probability.

These articles discuss particular mathematical topics within each content area. The problems and student solution strategies that are included illustrate the nature of the understanding that is expected about these topics. The examples of the children's work also indicate the depth of achievement that is possible from students in kindergarten through middle school.

1.

NUMBER AND OPERATIONS; ALGEBRA

THE two content standards Number and Operations, and Algebra are intimately linked, and the articles in this section illustrate how algebraic processes have their roots in elementary arithmetic activities. The following six articles investigate three strands of mathematical competence—number and computational fluency, algebra as generalized arithmetic, and algebra as the study of patterns and functions.

Number and Operations

INSTRUCTIONAL programs from prekindergarten through grade 12 should enable all students to —

- *understand numbers, ways of representing numbers, relationships among numbers, and number systems;*
- *understand meanings of operations and how they relate to one another;*
- *compute fluently and make reasonable estimates.*

The short commentary by Calvert sets the stage for the other articles in this section. Calvert's concern is with students' lack of number sense rather than their overdependence on technology. She stresses the importance of developing students' abilities to work flexibly with number operations, use mental calculations, and recognize and use relations among numbers and operations.

The other two articles describe important aspects of the development of number and computational proficiency across the elementary grades. Isaacs and Carroll describe a sequenced strategies-based approach to basic-facts instruction. This type of instruction builds on primary children's informal mathematical knowledge, encouraging them to use personally meaningful strategies for solving problems involving simple number combinations. Basic facts eventually become automated; however, this knowledge is the result of meaningful engagement with numerical contexts rather than of rote memorization.

The article by Curcio and Schwartz uses a series of short vignettes of children in grades K–6 to describe the development of arithmetic algorithms. Formalization is built on children's previous work with invented strategies, occurring only after children have had time to explore mathematical relationships and work with their own invented algorithms. A key element in this instructional approach is listening and responding to the computational understanding of each child.

A Dependence on Technology and Algorithms or a Lack of Number Sense?

by Lynn M. Gordon Calvert

"My students stop thinking when they are given a calculator. They just punch in numbers to get an answer. It doesn't take long before they are dependent on their calculators for even the simplest questions."

Sentiments that are similar to those above have been stated by many of the preservice and practicing teachers with whom I work. As a former middle school teacher, I understand their frustration; I, too, was often dismayed whenever I saw my students "punching in" such expressions as –9 + 4. When I suggest that the calculator does not cause the dependence, my argument is often countered with classroom and real-life experiences that appear to indicate the opposite. An example similar to the following has frequently been given as proof of the unacceptable dependence that occurs whenever technology is used for calculations:

A customer brings an item to the cash register, and the clerk rings in a purchase for $3.28. The customer hands the clerk a $5.00 bill. After the register calculates $1.72 as the amount of change to be returned, the customer digs into his pocket and finds 3 cents and places the change on the counter. At this point the store clerk is at a loss for what change to give.

Most of us have experienced this situation from one side of the cash register or the other. Although I agree that it is a problem, technology neither causes this dependence nor lies at the root of the problem. The counterexample that I provide for this scenario is the procedure used in places where change for purchases is still determined manually. Most clerks do not mentally calculate $5.00 – $3.28. Instead, they use a counting-up strategy or algorithm: start with $3.28; adding two pennies makes $3.30; adding two dimes makes $3.50; adding two quarters makes $4.00; and adding one dollar makes $5.00. If the three cents is introduced at any time during this algorithm, the confusion is the same as if technology was used. Whether the change is calculated mechanically or manually, the clerk frequently either ignores the three cents altogether and returns it with the original change or searches for coin trades afterward to reduce the number of coins. Recognizing in the midst of using technology or an algorithm that the change required for a purchase of $3.28 from $5.03 is the same as the change required for the simpler question of $3.25 from $5.00 demands a shift in thinking altogether, hinged on the ability to work flexibly with number operations.

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Dependence, if such a thing does occur, is not confined to technology. How often have we seen a child (or an adult) use a paper-and-pencil algorithm to "punch in numbers to get answers," as in the following computation?

As practicing teachers, we have often encouraged and enforced what appears here as a dependence on a standard algorithm by having children practice a procedure repeatedly, usually with very little thought or understanding. Rather than use a paper-and-pencil algorithm, a more reasonable approach to this question may be to calculate mentally that 10,000 minus 100 is 9,900, plus 1 is 9,901.

Continuing to place blame on technology or even algorithms for causing dependent behavior steers us

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away from looking at a much deeper and more difficult issue. Many children and adults lack the facility to recognize and work with relationships in and between numbers and number operations. The deeper issue at stake involves the development of number sense.

The examples discussed here show inflexibility in adjusting one's thinking during computation. Regardless of whether calculators are available, as teachers we need to ask questions that emphasize relationships and promote a mental flexibility with numbers. For example, rather than ask first-grade students a series of unrelated single-digit-addition equations, we could ask such questions as "If you have six cookies and two plates, in how many ways can you arrange the cookies?" Such tools as concrete objects, paper for drawing, and even calculators should be provided. Such questions could encourage the development of numerical relationships as children begin to notice physically and symbolically that adding one cookie to one plate requires taking one cookie away from the other plate (1 + 5 = 6; 2 + 4 = 6).

In third grade, rather than emphasize the isolated memorization of multiplication facts, encourage students to create mental images of these facts and to think about their relationships to other known facts, such as those shown in the array in figure 20.1. Here, 6 × 4 can also be thought of as 5 × 4 + 4 or as 3 × 4 doubled.

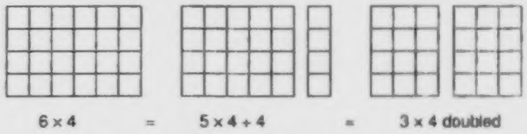

Fig. 20.1. 6 × 4 number relationships

By reducing the number of isolated arithmetic questions that we ask and by increasing the number of problems that emphasize relationships, perhaps we can dispel the myth that technology directly hinders learning basic facts and leads to dependence. With such a vision, technology can actually be used to assist and promote the development of number sense by helping children develop such skills as counting, estimating, noticing patterns, and using guess-and-check problem-solving strategies, as well as helping them develop a reflective need to judge the reasonableness of their answers. Also, the use of technology and algorithms in combination with number-sense relationships could give students more efficient ways to calculate answers. For example, if a problem situation involved buying 11 plastic toys for 78 cents each, a student may choose to calculate mentally that 10 × 0.78 is $7.80, then use a calculator or pencil and paper to add $7.80 and 0.78 for a total of $8.58.

Ample time and many experiences are required to develop number-sense relationships. Focusing on the belief that technology and algorithms cause dependence is misguided. Keeping calculators out of the elementary classroom may remove the appearance of dependence on technology, but the root of the problem remains intact and an important tool for learning and doing mathematics is lost. We need to continue to focus our energies on finding ways to promote children's mental facility with number concepts and operations so that our students are able to apply or invent flexible and efficient computation methods as situations warrant.

Strategies for Basic Facts Instruction

by Andrew C. Isaacs and William M. Carroll

In a class of beginning second graders, children explained with these typical replies how they solved the number fact 8 + 7: "I know 7 + 7 is 14, and 1 more is 15"; "8 + 2 makes 10. But 7 has 5 more, so the answer is 15"; and "I just knew the answer was 15." Teaching basic number facts like 8 + 7 has been a goal of elementary mathematics instruction for more than 100 years and continues to be important today.

Although most teachers agree that students' fact mastery is important, many are unclear about how to seek it in ways that are consistent with the NCTM's Standards (1989, 1991, 1995). They even disagree about what knowing the basic facts means and when, or even if, students should achieve mastery. Is it appropriate to expect first graders to memorize addition facts, or will this task interfere with their mathematical thinking? What classroom practices can build both understanding and quick recall? Can fact mastery be achieved through problem-solving activities, or is practice necessary? If current reforms in mathematics education are to succeed, questions about the basic facts need answers.

Why Should Children Learn the Facts?

Most people recognize that children should learn the basic facts because knowing them is useful, both in school and in life out of school. Estimation and mental computation require the use of basic facts. How can students use 80 × 40 to estimate 84 × 41 if they do not know 8 × 4? Students who know their facts do better in school mathematics. Parents, teachers, and the public expect schools to teach the basic facts.

Fortunately, no conflict need exist between fact mastery and school mathematics reform. Many goals of reform—helping students make connections between school mathematics and the real world, helping students develop conceptual understanding as well as procedural skills, helping students learn to explain their thinking and to understand others' explanations—can be achieved through a program that also leads to fact mastery. Properly approached, the basic facts offer excellent opportunities for teaching children to think mathematically.

How Should the Facts Be Taught?

The traditional rote approach to the basic facts, with frequent drill and timed tests, has serious disadvantages. Premature demands for quick performance can induce anxiety and undermine understanding. Rigid schedules for mastery do not accommodate individual differences and have unfortunate outcomes for some children. The rote approach encourages students to believe that mathematics is more memorizing than thinking.

Today, however, the outlines of a better approach are clear. This approach begins with children's natural thinking. The essence of many current reforms in primary-grade mathematics, including the approach to the basic facts described here, is to recognize and build on the wealth of informal mathematical knowledge that children bring to school. Traditionally, much of this knowledge has been ignored or suppressed.

Early work with basic facts should help children refine and extend their natural strategies for solving simple problems. As children increase their proficiency

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at various strategies, they begin to remember the simplest facts. Knowing the simpler facts makes possible more efficient strategies for harder facts. Gradually, students master more and more efficient strategies and commit more and more facts to memory. At the end of the process, students can accurately and automatically produce all the basic number combinations. Many of these combinations are recalled from memory, but a few may also be found through quickly executed strategies or suitable rules. Fact strategies and recall are used by both children and adults. We know a research mathematician, for example, who solves 8 × 7 by doubling 7 three times—14, 28, 56—but he does the doubling so quickly and effortlessly that it is automatic.

In this section of the article, we describe this "strategies" approach for the addition and subtraction facts; many of the same ideas also apply to the multiplication and division facts. First we describe how children's informal knowledge, especially their knowledge of counting and of part-whole relationships, can be used in beginning fact work. Then we describe how children use facts that they know to derive facts that they do not know. Finally, we discuss the role of practice and sketch a possible sequence for addition- and subtraction-facts instruction.

Counting to solve problems

Perhaps the best way to extend primary-grade children's informal understanding of addition and subtraction is by asking them to solve simple problems—without telling them how those problems are to be solved. These problems can come from real life, classroom situations, textbooks, the teacher, or the children's imaginations: "An adult movie ticket is $7 and a child's ticket is $4. How much for one adult and one child?" "Miriam wants a toy horse that costs $15. She has $8. How much more does she need?"

As students encounter the problems, they should be encouraged to devise their own solution procedures by looking for patterns, thinking logically, and using manipulatives. The adult approach—reducing such problems to addition or subtraction number sentences and retrieving the answers from memory—is not a natural strategy for young children. Instead, primary-grade children tend to use direct modeling, counting, and derived-fact strategies (Bergeron and Herscovics 1990).

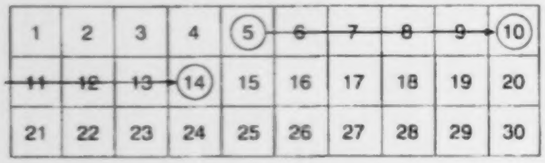
Direct-modeling techniques are generally the first to appear for a given type of problem. The child counts out objects to represent the quantities in a problem, performs actions with the objects that parallel the problem situation, and counts some set to find the answer. To solve the preceding movie-ticket problem, for example, a child might count out seven chips for the cost of the adult's ticket and four chips for the child's ticket. Then by counting all the chips, the child can find the total cost. Direct modeling is used by young students to solve simple addition and subtraction problems and even some surprisingly difficult multiplication and division problems (Carpenter et al. 1993).

Direct modeling can be rather inefficient, however, especially for problems with larger numbers. Eventually, direct-modeling strategies are supplanted by oral or mental counting strategies. A large number of such strategies for addition have been identified (Resnick 1983; Carpenter and Moser 1984; Baroody and Ginsburg 1986; Siegler and Jenkins 1989). Two common strategies are "counting all" and "counting on from the larger addend."

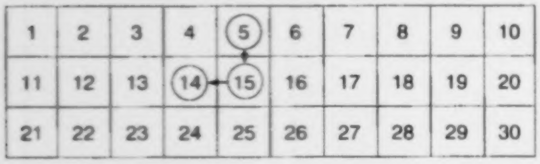
A significant feature of most counting strategies is that the child must keep track of how many numbers have been said. To solve 3 + 4 by counting all, the child first counts three numbers (1, 2, 3) and then four more numbers (4, 5, 6, 7). This "double counting" can be tricky; using fingers or objects can help. Similar counting strategies exist for subtraction, such as counting up from the smaller number to the larger and counting back from the larger number.

How best to help children advance to more efficient strategies is an open question. Certainly laying out a strict sequence of strategies and expecting all children to adhere to it would be ill-advised. Not only do different children progress at different rates, but the same child may use different strategies on different problems or even on the same problem in different contexts. However, if teachers hesitate too much to demonstrate better methods, students' progress may be impeded.

One approach to encourage more efficient methods is to ask children to share their strategies. This method helps them improve their communication skills and learn from one another. Figure 21.1, for example, shows how two first graders used a hundreds chart to find 5 + 9. In a typical class, children will use and describe various approaches, so most children will encounter new but understandable techniques. The teacher may also propose and model strategies, taking care that certain strategies do not become "official" while other strategies are discouraged. The teacher should not be disappointed when a child does not adopt more efficient strategies right away—development may be advancing below the surface at the rate best suited to the child.



"I started at 5. Then I counted 5 more to 10, then 4 more to 14."



"I started at 5 and jumped ahead 10 to 15. But it was only 9, so I moved back 1 to 14."

Fig. 21.1. Hundred-chart strategies for 9 + 5

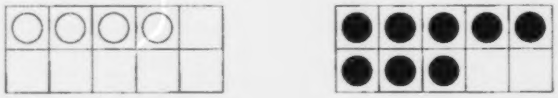
Class discussion of strategies should be supplemented with exercises designed to facilitate more sophisticated strategies. For example, the advance from counting all to counting on depends in part on skill in counting ahead the correct number of counts from another number. By practicing counting outside any problem context, children can develop competencies that support more sophisticated problem-solving strategies. A variety of such exercises should be included: counting forward and backward by 1's, starting at various numbers; skip counting, especially by 2's, 5's, and 10's; and counting forward and backward a given number of numbers: "Start at 8 and count forward 3" or "Start at 11 and count back 2," and so on.

Parts and wholes

Another central understanding that young children bring to school is that a quantity can be broken into parts that taken together equal the original quantity. They also understand that if they have some and get more, then they end up with more; and if they have some and lose some, then they end up with less (Resnick, Lesgold, and Bill 1990). Developing these basic "parts and whole" ideas further is essential to understanding addition and subtraction.

Ten-frames, like those in figure 21.2, are good for developing part-whole understandings involving the landmark numbers 5 and 10 (Thompson and Van de Walle 1984; Thornton and Smith 1988; Van de Walle 1994). These understandings are especially useful in addition- and subtraction-fact work. For example, the ten-frame for 8 in figure 21.2 shows that 8 is 3 more than 5 and also 2 less than 10. The ten-frame for 4 shows that 4 is 1 less than 5 and 6 less than 10. Once students learn facts involving 5 and 10, especially the pairs of numbers that sum to 10, they can use their knowledge to solve other basic-fact problems.

4 + 8 = ______



Maria made 4 and 8 on the ten-frames. Then she moved 2 counters to make 10 and 2 more to make 12. So 4 + 8 = 12.

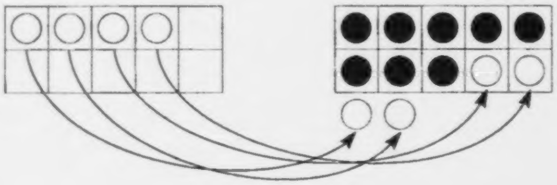


Fig. 21.2. Ten-frames showing that 4 + 8 = 12

Derived facts

Although most young children do not have automatic command of the basic facts, most adults do. In between is a stage in which some facts are known and others are not. During this stage, many children use the facts that they know to derive the facts that they do not know. Class discussion of such derived-fact strategies helps students learn from their peers and also legitimizes the use of strategies, thus encouraging the invention of further strategies (Steinberg 1985). Class discussion should examine the relative advantages of different strategies for various problems (Thornton and Smith 1988). Encouraging the discussion of multiple solutions enhances strategy and fact knowledge and helps students develop methods for mental and multidigit computation. Instruction to facilitate specific strategies can also be worthwhile.

The "doubles" facts are often useful for deriving unknown facts. For example, a child might solve 3 + 4 by noting that 3 + 3 = 6, so 3 + 4 must be 1 more than 6. Facts like 8 + 6 can be solved either by "sharing" (8 + 6 = 7 + 7 = 14) or by using a double and adding 2 more (8 + 6 = 6 + 6 + 2 = 12 + 2 = 14). Since doubles-based strategies are common, care should be taken that children learn the doubles facts early. Many games can be modified so that they involve doubles. For example, games with two dice can be played with one die doubled instead. A chart with examples of addition doubles, such as 6 eggs + 6 eggs = 12 eggs, can be kept as a class project and explored for patterns, for example, all the sums are even. Brief oral drills are also appropriate as children are consolidating their knowledge of these facts.

Many other common strategies involve 10. For example, a child using 10 might solve 9 + 7 in several steps: 9 + 1 = 10 and 10 + 6 = 16, so 9 + (1 + 6) = 16. To support such strategies, early attention should be given to complements of 10, such as 6 + 4, 7 + 3, and so on. The ten-frame activities described previously are ideal.

Children also use derived-fact strategies for subtraction. Some subtraction strategies are refinements in counting, such as using 10 as a bridge in counting up or down. For example, to solve 13 – 6, count up 4 from 6 to 10, and then up 3 more from 10 to 13, for a total counted of 4 + 3 = 7. Other strategies involve using known addition facts to derive unknown subtraction facts: 15 – 8 = 7, since 7 + 8 = 15.

Practice

The place of practice in school mathematics is much disputed. We think that a reasonable position was described by William Brownell more than fifty years ago. Brownell and his student Charlotte Chazal found that under certain conditions, practice can be harmful. Premature demands for speed, for example, caused many children simply to become quicker at immature approaches. Delaying drill was found to result in better understanding and ultimately in less need for drill (Brownell and Chazal 1935). Over the years, unfortunately, some educators have misunderstood this and similar research and have concluded that all practice is bad.

We believe that the right conclusion is that premature practice can be detrimental but that properly managed practice is essential in the development of expertise—whether the subject is piano, tennis, or the basic facts (Brownell 1956; Chase and Chi 1981; Siegler 1988; Anderson, Reder, and Simon 1996). Brief, engaging, and purposeful practice distributed over time is usually most effective. Problem solving is one important source for such practice, but games, computers, or even old-fashioned technology like flash cards and choral drills can also be useful.

An instructional sequence

The preceding ideas can be used to sketch a possible instructional sequence for the addition and subtraction facts. Note that in this sequence, the facts are grouped by strategy rather than by sum. A double like 6 + 6, for example, may be easier than a problem like 4 + 3 and, accordingly, appears earlier in this sequence.

1. Basic concepts of addition; direct modeling and "counting all" for addition
2. The 0 and 1 addition facts; "counting on"; adding 2
3. Doubles (6 + 6, 8 + 8, etc.)
4. Complements of 10 (9 + 1, 8 + 2, etc.)
5. Basic concepts of subtraction; direct modeling for subtraction
6. Easy subtraction facts (– 0, – 1, and – 2 facts); "counting back" to subtract
7. Harder addition facts; derived-fact strategies for addition (near doubles, over-10 facts)
8. "Counting up" to subtract
9. Harder subtraction facts; derived-fact strategies for subtraction (using addition facts, over-10 facts)

How Can Fact Knowledge Be Assessed?

The assessment of children's fact knowledge should be balanced, based on multiple indicators, and aligned with instruction. Assessment should help the teacher evaluate not only answers but also how students are getting those answers and whether students understand the underlying mathematical concepts and connections. For example, a student might appear to know the basic facts during problem-solving activities but actually be relying on counting. Another student might be quite proficient on isolated facts but have a weak grasp of the concepts of the operations. A combination of assessment techniques can clarify each student's strengths and weaknesses and can help the teacher plan instruction.

What is meant by fact proficiency differs by age. For example, the first grader in figure 21.3 used finger counting, doubles, and recall in answering various

Student: [Student reads] Three plus five equals . . . . Hm. [Pause] Three. [Student then counts on fingers, putting up five fingers at one time.] Four, five six, seven, eight. Eight.

Teacher: How did you figure that out?

Student: I did it on my fingers.

[Child is shown card with 5 + 5 on it.]

Student: [Rapidly] Five plus five equals ten.

Teacher: How did you get that?

Student: I figured it out in my mind.

Teacher: You always knew it? [Student indicates yes.] Okay, what's six plus six?

Student: [Fairly rapidly] Thirteen.

Teacher: How did you get that?

Student: Because I counted five and then added two more: five plus five and two more.

Teacher: And you got what, thirteen?

Student: Yeah. [Child reads next problem] "Seven plus nine equals. . . . " [Pause. Then child begins to count on fingers. First, child apparently begins to count all—to seven on one hand. Then starts over, saying seven and starting over on fingers, putting up nine fingers one at a time.] Eight, nine, ten, . . . , sixteen. Sixteen.

Teacher: Sixteen. Okay, so here's another one. If seven plus nine is sixteen, what's nine plus seven?

Student: [Two seconds, then child responds with enthusiasm.] Sixteen!

Teacher: How do you know that?

Student: It doesn't matter which one's first. But they're always, . . . they're just always like, . . . no matter what is first, they're always the same number.

Teacher: Here's another one. Four plus ten.

Student: [Quickly] Four plus ten is fourteen.

Teacher: How did you get that?

Student: Well, I just figured it out.

Teacher: On your fingers.

Student: No. I thought.

Fig 21.3. A "reasonably proficient" first-grade student

facts. These responses show a good range of mathematical understanding and indicate that the student is reasonably proficient in the basic addition facts. By third or fourth grade, however, we would expect all addition facts to be answered quickly by recall or automatic strategies.

Samples of students' work

Collecting samples of student work is a good way to gather evidence about students' knowledge and application of facts. These performance-based samples should come from activities in which students use facts. For example, figure 21.4 shows a number-collection box in which a second grader recorded different ways of making a target number, in this example, 9. Such exercises help children develop their understanding of addition and subtraction and also afford opportunities for assessing fact knowledge.

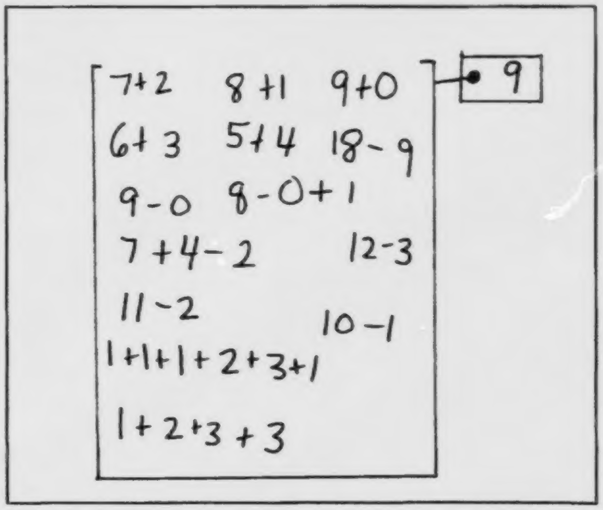


Fig. 21.4. "Number -collection box" for 9

Although performance-based samples offer evidence of conceptual understanding and applications, information about students' level of proficiency is often limited. Typically, for example, work samples do not reveal whether the student used counting, derived-fact strategies, or recall. The information is also limited to the particular numbers involved in the sample. Without additional information, a teacher might find it difficult to plan meaningful instruction.

Observations, class discussion, and interviews

Observing students engaged in games and problem-solving activities can yield rich information about their fact knowledge. For example, as students play a game, a teacher may notice whether they are using counting strategies, derived-fact strategies, or known facts. More important, the teacher can get a better idea of the range of students' knowledge with individual facts. Brief observational notes can help with planning individual and whole-class instruction: Tomás is still counting on, even for the easy facts; Juanita knows most of the double facts, and she also uses these facts to solve some of the near doubles. Useful information can also be obtained during class discussions as individuals explain their solutions to story problems or other problem-solving activities.

Short individual interviews are probably the best way to get a full picture of a student's progress with basic facts. Although these interviews are time-consuming, with a little planning a teacher can manage a five-minute interview twice a year with each student, perhaps spacing the interviews over a month. More frequent interviews with students who are having difficulties can help pinpoint problems.

Inventory tests

Although clearly an overreliance on timed tests is more harmful than beneficial (Burns 1995), this fact has sometimes been misinterpreted as meaning that they should never be used. On the contrary, if we wish to assess fact proficiency, time is important. Timed tests also serve the important purpose of communicating to students and parents that basic-fact proficiency is an explicit goal of the mathematics program. However, daily, or even weekly or monthly, timed tests are unnecessary.

An inventory test on all the addition and subtraction facts might be done at the beginning of second and third grades. These tests establish a baseline for measuring progress and provide information that can be useful in planning instruction. End-of-the-year tests, and perhaps mid-year tests, can be used to document progress. Similar inventories for multiplication and division facts might be given in fourth and fifth grades. We recommend against any timed tests during first grade, or any frequent use in the primary grades, because they work against a strategies approach to the facts. That is, in a timed situation, students will be less likely to explore the more sophisticated strategies necessary to make progress.

Small-scale diagnostic tests

Although positive inventory-test results are reassuring, they yield limited information. It may be, for example, that a student is proficient at some facts but uses counting for other facts. For this reason, it is helpful to test smaller sets of facts with short diagnostic tests linked to specific strategies. For example, after students work on the doubles, a quick test of the doubles facts can indicate whether students are ready to move on. As students move toward proficiency, short tests of mixed fact strategies—doubles, near doubles, and complements of 10—can also be useful for diagnostic purposes. A three-second rule is often used as a benchmark of automaticity (Van de Walle 1994), although some teachers prefer two seconds (Thornton 1990). Note that these criteria allow enough time for students to use efficient strategies or rules for some facts.

If we expect students to move from counting strategies toward facility with facts, then an occasional low-stress test or practice is consistent with our goals and the message that we want students to receive. The crucial point is to emphasize individual progress. In kindergarten and first grade, counting strategies are appropriate for solving the basic addition facts, but we should have concerns about students who are still "counting all" in the middle of second grade or who have not mastered even the easiest addition facts. Not diagnosing these students' difficulties and planning appropriate instruction for them does them a disservice. A balanced approach to assessment—work samples, some observations, some test information, and some interview information—gives the teacher, the student, and the parent a more complete portrait of the child's fact knowledge, how it is connected to other mathematical knowledge, and what progress is being made.

Conclusion

A strategies-based approach to the basic facts has several advantages. First of all, it works: children do learn their facts. Rathmell (1978) found that teaching children thinking strategies facilitates their learning and retention of basic facts. More recent studies have confirmed this effect again and again. These findings should not be surprising: a strategies approach helps students organize the facts in a meaningful network so that they are more easily remembered and accessed. Further, although many facts become automatic, adults also use strategies and rules for certain facts. Many strategies, such as properties of the multiples of 9, both support facts and supply links to other mathematical concepts, such as divisibility. Many researchers have recommended strategies-based approaches for learning the basic facts, including Thornton (1978, 1990), Cook and Dossey (1982), Myren (1996), and Chambers (1996).

A strategies-based approach also builds students' understanding and confidence. De-emphasizing rote memorization encourages students to use their common sense in mathematics, thus supporting concept development. International research confirms that early fact automaticity and problem solving are not discrepant goals (Fuson, Stigler, and Bartsch 1988; Stigler, Lee, and Stevenson 1990). The cost in instructional time is also low: delayed practice often means less practice. Children's success at learning their facts also reassures parents about their children's mathematics program.

Certain pitfalls must be avoided in a strategies-based approach. One danger is that children might learn strategies by rote, so that mindless memorization is replaced by equally mindless "strategies" (Cobb 1985). Another possibility is that class discussion might degenerate into the tedious recitation of every imaginable method, with little critical appraisal of the various approaches. Encouraging multiple ways to solve fact problems may also lead students to conclude that memorizing the facts is not important. We believe, however, that in most situations a thoughtful and sensitive teacher can avoid these hazards.

Our purpose has been to address important questions about the basic facts, for fear that neglecting them will undermine reforms now under way. We worry that our efforts to correct for a narrow focus on lower-level skills will lead to an overcorrection. We recall Brownell's warning at the beginning of the New Math era: "In objecting to the emphasis on drill prevalent not so long ago, we may have failed to point out that practice for proficiency in skills has its place too" (1956). We must remember that successful education involves both basic skills and higher-order processes.

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There Are No Algorithms for Teaching Algorithms

by Frances R. Curcio and Sydney L. Schwartz

WHAT comes to mind when we hear the word *algorithm?* We may think of it as a procedure, an efficient method, or a rule for computation. From our own formal school experiences, we may recall pages of skill drill and practice, or lengthy efforts to accomplish rote memorization. Those of us fortunate enough to have experienced and understood the power of an algorithm may recognize its usefulness as a problem-solving tool. For those of us who have learned algorithms without meaning and understanding, practicing algorithms reinforces the view that the imposition of methods, rules, and procedures will ensure the accuracy and efficiency of computation and symbol manipulation.

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The instructional implications of the research literature on learning related to acquiring conceptual/meaningful knowledge and procedural/mechanical knowledge can be described from several different perspectives. From one point of view, teachers are advised to build understandings *before* presenting an algorithm (Brownell 1987; Driscoll 1981; Fehr 1988). The intent is to balance the logic of the algorithm with its use, recognizing that one without the other leaves the learner at a disadvantage. In contrast, another position suggests that the relationship between conceptual and procedural knowledge is so complex that for the most part, developing conceptual knowledge does not guarantee the acquisition of related procedures (Baroody and Ginsburg 1986; Silver 1986). From this perspective, designing instructional tasks to support the development of conceptual and procedural knowledge in tandem is a challenge. From still another point of view, we as teachers are advised to suspend the "show and tell" method so that children may invent their own rules and algorithms (Kamii and Lewis 1993). Those taking this approach would expect that children will draw on their prior experiences and understandings to create and construct their own rules through the processes of observing and analyzing mathematical relationships.

What we as teachers share with these approaches is a concern that students develop an understanding of a core of necessary and useful algorithms in a timely way. We also share with these approaches a belief that understanding an algorithm, either its logic or the sequence of the procedural steps, leads to a more secure mastery than simple rote memorization.

Since time in school is insufficient for students to build *all* the mathematical algorithms required to progress through the mathematics curriculum outlined for the school years, it seems wise to find an approach that balances invention with the development of traditional and alternative algorithms to foster students' understanding. In the interest of balance, students need opportunities to experience the inventiveness of mathematics—getting a feel for creating a method, rule, or procedure by discussing and analyzing patterns and relationships, recognizing consistencies, and formulating conjectures and generalizations. In this way, they can develop and appreciate the power of mathematical thinking at each new level of their understanding.

Algorithms have an important place in the mathematics curriculum. Consequently, issues surrounding the teaching of algorithms focus not on whether to teach them but rather on balancing and connecting the development of algorithmic thinking involving inventions with the more traditional and standard algorithms that permeate the curriculum. Most of the strategies for teaching conventional algorithms tend to be traditional, shaped by years of practice and guided by widely used teacher manuals accompanying school-mathematics-program textbooks and workbooks. We are much more tentative in using strategies that support the process of generating initial rules of relationship leading to fully defined and useful algorithms. Our knowledge and understanding of the mathematics embedded in an algorithm and our knowledge about how algorithmic thinking manifests itself in the developmental process drive our instructional decision making.

Building Instruction Based on Children's Inventions

We began our collaboration with teachers with the agreed-on belief that children are natural inventors and creators of original solutions (Davis 1984). The many documentations of children learning through "play" and their own interactions with materials leaves little doubt that they are capable of inventing rules of relationship before any formal instruction (Kamii 1985, 1989). In addition, we have learned in our work that children may also discover relationships that are unknown to the teacher (Curcio and Stewart 1992). Their inventions and creations or original solutions flow from confronting problems that engage their minds, with or without manipulating variables represented by concrete materials or abstract symbols.

The vignettes that follow present examples of children at various stages in their development of algorithmic thinking. In each example, the teacher listened to the student, posed questions to elicit his or her level of understanding, and made instructional decisions based on the teacher's knowledge of the students.

Following a young child's lead

In the first vignette, Eli, a five-year-old youngster at the entry level of discovering relationships among numbers, generates initial rules about how to operate with numbers.

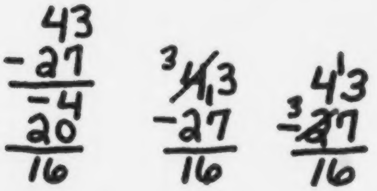
> On a kindergarten school trip, Eli read a numbered street sign and asked his teacher, "What is two-three?" The teacher responded, "What do you mean? Do you want me to tell you what the number two-three is, or do you want to know how much is two and three more?" Eli paused for a while and then asked, "How much is two and three?" The teacher held up two fingers on one hand and three on the other, inviting him to count. He counted five. At the next corner, he asked, "What is two and four?" The teacher repeated her question, "Do you want to know what number is two-four or how much is two and four more?" Again he paused, and again, asked, "How much is two and four?" The teacher repeated the procedure of holding up fingers, and he counted six. At the next corner, he asked, "What is two and five?" This time the teacher said, "Two and three is five, two and four is six. How much do you think two and five is?" He paused for a longer period than previously and then said, "Seven." When the teacher asked him how he had figured it out, he said, "Three–five, four–six, five–seven…"

In this vignette, Eli created his own way of thinking about rules for combining numbers. He shared a beginning understanding of the algorithm about incrementing quantities by one. At this point, the teacher had the choice of reaffirming the addition algorithm that Eli had started to develop or extending his experience with the algorithm so that he could continue to test it. Because he had just begun to move his understanding of the algorithm from the intuitive to the conscious level and use it to answer his own questions, the teacher followed the child's lead as he continued to figure out how much is two and six, two and seven, and finally two and eight.

Building on divergent thinking

In this next vignette, Jackie, a third grader who had already had experiences with subtraction, used an abstraction of the meaning of the minuend and subtrahend to create her own method of subtracting quantities.

> Jackie's teacher gave her the following problem: "Jackie has 43 stickers. After she gives her best friend Rosa 27 stickers, how many stickers will Jackie have left?" Jackie explained that she needed to take away 27 stickers from the 43 stickers. Her solution to the problem is shown in figure 22.1a.



(a) Jackie's method

(b) Decomposition method

(c) Equal-additions method

Fig. 22.1. Subtraction algorithms

Many times when children are taught a traditional algorithm, they are told, "You cannot subtract a larger number from a smaller number." Jackie did just this—in a process similar to "Kye's method" (Davis 1984, pp. 18–19), she subtracted 7 from 3, yielding –4. Then she subtracted 20 from 40, yielding 20. Finally she added –4 + 20 for her answer of 16. The teacher was surprised that Jackie had solved the problem with such a sophisticated method. She challenged Jackie to explain to the class how she thought about such subtraction examples. Jackie seemed to have an understanding of negative numbers, and she seemed to be applying her understanding in an abstract situation. The teacher related Jackie's method to various other subtraction algorithms, including the traditional decomposition method of subtraction (see fig. 22.1b) and the equal-additions algorithm (see fig. 22.1c; Spitler 1979); some teachers may choose to present and discuss culturally based alternative algorithms (see Philipp [1996]). When asked which method she preferred, Jackie indicated that her own way was the best for her. The teacher supported her preference.

In the next vignette, during the process of analyzing the data in a table, Garin, a fifth grader, discovered a relationship that at first baffled his teacher.

> When Garin's class was experimenting with "Growing Squares" in Mouse and Elephant (Shroyer and Fitzgerald 1986), his teacher asked the class to examine the relationship between the perimeter and the area of various squares (Curcio and Stewart 1992). The students built a table (see fig. 22.2), described the patterns that they saw, and verbalized the relationships between the measure of the edge of a square and its perimeter ($4n = p$) and between the measure of the side of a square and its area ($n^2 = a$). After a very rich discussion about the relationships, Garin approached the teacher to explain another, "more complicated" pattern that he saw (see fig. 22.3).

As the teacher and her colleagues analyzed what Garin had done, she realized that his discovery could be expressed as an algebraic identity: $n(n - 4) + 4n = n^2$. Developing algorithmic thinking is closely related to developing algebraic thinking because it requires pattern recognition and the representation and generalization of relationships. Garin's teacher had the choice

Edge	Perimeter	Area
1	4	1
2	8	4
3	12	9
4	16	16
5	20	25
6	24	36
7	28	49
.	.	.
.	.	.
.	.	.
n	$4n$	n^2

Fig. 22.2. Students' table of perimeters and areas.

Edge	Perimeter		Area
1 × −3 +	4	=	1
2 × −2 +	8	=	4
3 × −1 +	12	=	9
4 × 0 +	16	=	16
5 × 1 +	20	=	25
6 × 2 +	24	=	36
7 × 3 +	28	=	49

Fig. 22.3. Garin's pattern

of approaching the task either deductively—by giving formulas for perimeter and area and then having the students practice with some examples—or inductively—by giving cases and having the students analyze and generalize from them. She chose an instructional strategy that builds on children's interpretation and analysis.

Probing for clarification

In the last vignette, Jessika, a sixth grader, used manipulative materials to analyze the relationship between equivalent fractions. She seemingly created a method for finding equivalent fractions, but her teacher realized that Jessika's reasoning needed to be tested against other examples.

> Jessika was in a grade-6 class working on equivalent fractions. Using fraction strips for halves, thirds, fourths, fifths, sixths, eighths, tenths, and twelfths, Jessika was able to identify three equivalent fractions: 1/2, 2/4, and 4/8. When asked to generate two more equivalent fractions, she wrote 8/16 and 16/32. The teacher was very pleased with her responses. However, when asked to explain how she had obtained her other equivalent fractions, Jessika said, "All I have to do is add the top number to itself and add the bottom number to itself and I'll get an equivalent fraction." Concerned about this way of thinking about equivalent fractions, the teacher asked, "Do you think 3/6 is equivalent to 1/2?" When Jessika responded affirmatively, the teacher queried, "Well, how does 3/6 fit your rule?" Jessika was perplexed. Her teacher asked her to keep thinking about this question. The teacher made a note to revisit this idea with Jessika in the next few days.

Sometimes a student discovers or develops an algorithm that may yield a seemingly correct answer, but the reasoning behind it is problematic. Although Jessika's results were correct, she was unable to connect the additive rule that she had developed (i.e., $1/2 = (1 + 1)/(2 + 2)$) with the meaning of the multiplicative method for generating equivalent fractions (e.g., $1/2 \times 2/2 = 2/4$). Unless she comes to terms with this method herself, it will be very hard for her to think of equivalent fractions multiplicatively. Here is an example where inventing an algorithm by focusing on an obvious pattern may work against the student unless the teacher has a deep understanding of mathematics and is capable of posing questions and designing tasks to get the student on the correct track.

Closing Comments

These vignettes illustrate the value of giving children opportunities to construct, discuss, modify, and compare algorithms. These children, who created their own ways of operating and manipulating, revealed to us their discoveries about mathematical relationships.

Although these interactions with children reaffirm our belief that children are active inventors of rules of relationships, we still struggle with deciding when and how to formalize an algorithm. It seems clear that several factors must be considered. One factor may be the extent to which the child has fully explored the mathematical relationship. Emerging discoveries need time before formalization. A second factor may be the frequency of the child's use of an invented algorithm. Formalizing such algorithms may set the context for the child to move on to exploring more complex relationships. A third factor may be the range of useful algorithms that the child already possesses. At this point, we may consider presenting an algorithm that is closely related to the child's existing store of algorithms. Our most effective resource for making instructional decisions comes directly from the children as we observe and listen to them.

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Algebra

INSTRUCTIONAL programs from prekindergarten through grade 12 should enable all students to—

- *understand patterns, relations, and functions;*
- *represent and analyze mathematical situations and structures using algebraic symbols;*
- *use mathematical models to represent and understand quantitative relationships;*
- *analyze change in various contexts.*

The three articles in this section represent two approaches to the development of algebraic thinking. The first pair deals with algebra as generalized arithmetic, whereas the final article examines algebraic foundations from a functional approach. Both aspects are important and, as these articles show, draw their roots from the mathematical experiences of the elementary grades.

MacGregor and Stacey list five aspects of number knowledge that are essential for algebra learning (e.g., understanding equality and recognizing operations). For each, the authors describe how that particular aspect can be problematic for learners and suggest instructional strategies to promote more appropriate understanding. The second article builds on the first, examining in depth the way that the notion of equality develops over the elementary grades. The authors point out the distinction between the informal use of "=" to stand for "what comes next" and an understanding of equality as a mathematical relationship. The latter meaning is foundational for a student's future algebraic development.

Algebra also encompasses the study of patterns and relationships, and the foundations for this aspect of algebra should be laid in the elementary grades. Unfortunately, as Economopoulos points out, the use of patterning activities in the early grades may not lead to appropriate conceptual development. The author recommends that instruction help children come to see pattern as structure — to know that patterns are predictable and are made up of repeating units. Children should understand that it is not enough simply to identify patterns; they must also be able to provide evidence of why a particular pattern exists.

A Flying Start to Algebra

by Mollie MacGregor and Kaye Stacey

ALGEBRA learning has its roots in the early grades, when children notice regularities in the ways that numbers work. From this beginning, they develop a knowledge of the properties of numbers and operations, later generalized as algebra. We have conducted research over several years with middle school and high school students learning algebra (MacGregor and Stacey 1993, 1995, 1997; Stacey and MacGregor 1997a, 1997b).

One finding of particular interest to elementary teachers is the importance of students' understanding of numbers. Students find algebra hard to learn unless they have a good knowledge of number properties and basic operations. They need to have thought about the general effects of operations on numbers and not have focused their attention only on getting answers to computations. Moreover, they need to feel comfortable using large numbers, fractions, and decimals, so that they can recognize when a general relationship or rule applies to the whole range of numbers. Children's discoveries about how numbers work form the building blocks of generalized number knowledge, which is developed and expressed in later years as algebra.

In this article we show how number work in the elementary school can be extended to prepare students for algebra. We suggest some practical strategies, which focus on five aspects of number knowledge essential for algebra learning:

- Understanding equality
- Recognizing the operations
- Using a wide range of numbers
- Understanding important properties of numbers
- Describing patterns and functions

Teachers can easily incorporate these strategies into their existing programs. For example, it is often worthwhile to spend time on variants of one problem, encouraging children to find out what happens when different numbers are used or when a pattern is extended. Many more ways in which the K–6 curriculum can be enriched to prepare students for algebra can be found in the February 1997 issue of *Teaching Children Mathematics*, a focus issue on algebraic thinking.

Understanding Equivalence Relationships

The problem

The language of arithmetic focuses on answers. The language of algebra focuses on relationships. For example, compare the arithmetic statement

$$287 + 146 = 433$$

with a typical algebraic statement,

$$2(x + 1) = 2x + 2.$$

The arithmetic statement gives an answer, and the "=" sign indicates that this answer has been found. While they are learning arithmetic, children come to interpret the equals sign as meaning "work out the answer." For example, when they see the equation 4 + 7 = ___ , they write "the answer": 11. They see the left-hand side of the equation as the question and the

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right-hand side as the answer, which in their experience is a single number. This interpretation, if it persists, is a major obstacle to understanding how algebra is used.

In the algebraic statement $2(x + 1) = 2x + 2$, although $2(x + 1)$ can be interpreted by students as a question, $2x + 2$ does not look like an answer. Instead, we have a statement about equality; the two quantities are equal, and one of them can replace the other when required. Students must learn how to write equalities. They also must learn how to generate chains of logically equivalent statements of equality, such as

$$\begin{array}{ll} \text{if} & (8x - 3) \div 3 = 2x, \\ \text{then} & 8x - 3 = 6x, \\ \text{and so} & 8x - 6x - 3 = 6x - 6x; \\ \text{so} & 2x - 3 = 0, \\ \text{so} & 2x = 3, \\ \text{and so} & x = 1\frac{1}{2}. \end{array}$$

Beginning algebra students often find it hard to deal with this form of logical reasoning. It will be helpful if they have already seen and used statements of equality in arithmetic.

What should teachers do when students use the equals sign to link steps in a computation? When they write, for example, $5 + 23 = 28 \times 3 = 84$, they are writing exactly what they do on their calculators. So how can it be wrong? The calculator interprets the equals sign as "carry out the operation," but in formal mathematics the sign links two expressions that have exactly the same value. Some teachers explain these two different meanings and have students use another sign, perhaps one they have invented themselves, to link steps in their working. For example, they could use an arrow to mean "gives," as in $5 + 23 \rightarrow 28 \times 3 \rightarrow 84$, read as "5 plus 23 gives 28, which multiplied by 3 gives 84."

Some suggestions

- Have children write statements of equality with the "answer" first. For example, instead of $5 + 9 = n$, write $n = 5 + 9$.
- Use a geoboard and a piece of string 24 geoboard units long to make different shaped rectangles. Children will see that all their rectangles have the same perimeter, 24 units. They may also notice that the areas of the rectangles are different even though the perimeters are the same. The important focus of this activity should be writing many different statements of equality, for example,

$$24 = 2 + 10 + 2 + 10,$$
$$24 = 4 \times 6 = 6 + 6 + 6 + 6 = 12 + 12,$$
$$3 + 9 + 3 + 9 = 4 + 8 + 4 + 8,$$

and so on. Seeing the reasons behind such relationships requires a generalization about properties of numbers that is deeply algebraic. Referring to the fixed length of the string will help children understand the equality. Erna Yackel's article in the February 1997 issue of *Teaching Children Mathematics* presents similar activities and a discussion of their value for developing algebraic thinking (Yackel 1997). To make the geoboard activity more challenging for older students, the units on the board can be given a value other than 1. Try 0.2 or 1/2, for example.

- Have children write their answers in alternative ways. For example, instead of $5 + 9 = 14$, they can write $5 + 9 = 10 + 4$ or $5 + 9 = 2 \times 7$, and so on. From the earliest years, children can appreciate that many ways of expressing a number are possible. Teachers should encourage them to demonstrate this understanding. Figure 23.1 shows the work of two first graders, Daniel and Amy, when their class was asked to draw Santa's eight reindeer, count the legs, write a number sentence to show how they had counted, and then write other number sentences to show other ways. The statements in quo-

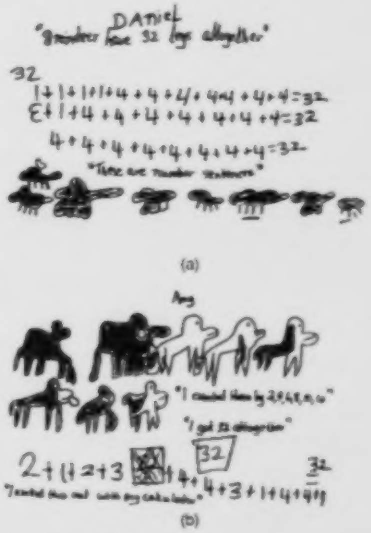
DANIEL
"8 reindeer have 32 legs altogether"
32
1+1+1+1+4+4+4+4+4+4+4=32
8+1+4+4+4+4+4+4+4=32
4+4+4+4+4+4+4+4=32
"These are number statements"
(a)
Amy
"I counted them by 2,4,6,8,10"
"I got 32 altogether"
32
2+1+2+3
4+4+3+1+4+4
32
"I worked this out with my calculator"
(b)

Fig. 23.1. First graders find different ways to count the reindeers' legs.

tation marks in figure 23.1 were put there by the teacher to record what the children said. Note that although Amy says she worked out her number sentence with a calculator, this way is not how she discovered it. She grouped the legs in different ways—ones, two, threes and fours—counted them, and then checked with the calculator.

Recognizing the Operations

The problem

The NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989, 91) states that students in grades 5–8 should have the opportunity to "understand how the basic arithmetic operations are related to one another." This understanding develops slowly and depends on first recognizing the four operations and how they are different. To solve many problems, in or out of school, it is often not important to know what operations are being used. In everyday life we frequently use methods based on counting, adding, doubling, and halving that can be carried out mentally. We break down a complex problem into small steps so that the solution becomes a collection of simple computations.

In contrast, to use algebra for solving a problem, the focus of attention is not on getting numerical answers to each step of the solution but on the operations used. Relationships and procedures must be recognized and made explicit. It is important, therefore, that students get experience in identifying which operation they are using to solve a problem. As stated in the NCTM's *Curriculum and Evaluation Standards* (p. 42), "connecting problem structures to operations should be emphasized throughout grades K–4 for both one-step and appropriate two-step problems."

One big obstacle to algebra learning in the middle years is a limited understanding of multiplication and division. Students who see multiplication only in terms of repeated addition can think of 3×4 as $4 + 4 + 4$ but are not able to make sense of $m \times 4$ because it does not tell how many 4s to add. Similarly, division needs to be understood as an operation in its own right—not just related to repeated subtraction. Children can be given problems that move them forward from using only primitive methods based on counting, adding, and subtracting. They need to become familiar with the concepts of multiplication and division that they will need for algebra.

Some suggestions

- Figure 23.2 shows four ways to solve the Cages problem, each method based on one of the four operations. Children should understand all these methods. It is important that they use them in a variety of contexts and learn to choose which one is most suitable for a particular problem. With numerical values, addition, subtraction, multiplication, or division can be used; but algebraically only one operation—division—will do. If the problem involves N birds, with x to each cage, the number of cages is N/x.
- To solve the Migrating Birds problems (see fig. 23.3), children will see that the first problem is easily solved without thinking about division; but for the second problem, division is much quicker than repeated addition or subtraction. For both problems, multiplication should be used to check the answers.

John needs new cages for his 96 canaries. He will put 6 canaries in each cage. How many cages does he need?

Repeated addition (use calculator)

$\boxed{6} + \boxed{6}\,\boxed{=}\,\boxed{=}\,\boxed{=}\cdots$ I added 6 sixteen times.

Repeated subtraction (use calculator)

$\boxed{9}\,\boxed{6} - \boxed{6}\,\boxed{=}\,\boxed{=}\,\boxed{=}\cdots$ I subtracted 6 sixteen times.

Trial multiplication and addition (mental reasoning)
I know that 10 sixes are 60 and that 5 sixes are 30; 60 + 30 = 90, that's 15 cages. It's almost enough. We need one more cage. That makes 16 cages.

Division (use calculator)

$\boxed{9}\,\boxed{6}\,\boxed{\div}\,\boxed{6} =$ The answer is 16.

Check that the answer "16 cages" is correct.
Use a calculator to multiply 16 by 6: $16 \times 6 = 96$. This equation tells me that if John has 16 cages and if each cage holds 6 canaries, he has 96 canaries altogether.

Fig. 23.2. "Cages": A problem that can be solved by using addition, subtraction, multiplication, or division

1. Every year when the weather in the Northern Hemisphere starts to get cold, small wading birds called *stints* and *sandpipers* set out from Alaska to fly to Australia. They have a long way to go, about 12,000 miles. Suppose that they travel 300 miles every day. How many days will the whole journey take?

   Multiplying and doubling (mentally)
   In one day they go 300 miles, so in 10 days they go 3,000 miles. Doubling that distance gives 20 days, 6,000 miles. Doubling again gives 40 days, 12,000 miles.

   Division (mentally or with calculator)
   Divide 12,000 by 300; 12,000 ÷ 300 = 40.

2. Some birds making the journey from Alaska to Australia take a longer route, about 13,500 miles. If these birds fly 230 miles every day, how many days will the whole journey take?

Note: Students may want to find out some of the amazing facts about migratory birds. The ones we refer to here are the red-necked stint, *Calidris ruficollis*, and the sharp-tailed sandpiper, *Calidris acuminata*. Many students in North America will have seen the Canada goose, a bird much larger than stints and sandpipers. How far is its annual migration? How fast does it fly?

Fig. 23.3. "Migrating birds": Two division problems

Using a Wide Range of Numbers

The problem

Some students enter the middle years having had very limited experience with numbers other than small whole numbers. They are not sure whether the procedures they have learned to use with these numbers can also be used for large numbers, decimal numbers, and fractions. This uncertainty is not surprising, because students learn to do written computations differently for different kinds of numbers. For example, adding fractions is not like adding decimals, as the following example shows.

Fractions

$$\frac{3}{4}+\frac{1}{2}=\frac{3}{4}+\frac{2}{4}=\frac{3+2}{4}=\frac{5}{4}$$

Decimals

$$\begin{array}{r} 0.75 \\ +\underline{0.50} \\ 1.25 \end{array}$$

When we talked with fourteen-year-old students, we noticed that they expected questions containing decimal numbers to be much harder than whole-number questions, or to require different rules. We had asked them to translate some statements into algebraic language. One statement was this:

If y is multiplied by 8, the result is 24.

Another was the following:

Three times y is equal to 0.051.

Students found the first question to be easy. They did not hesitate in writing correct equations, such as $8 \times y = 24$, or $8y = 24$. However, some of them were worried by the second question. They took much longer to translate it than they had taken for the whole-number question, although it has the same structure, $3 \times y = 0.051$, or $3y = 0.051$. Some students said that they had no idea what to do. Maria said immediately, "I'm not good at decimals." Larry looked at the question and said, "That number! I don't like those numbers, decimals." Ben said, "I don't think I know how to do this one."

Students like Maria, Larry, and Ben need to know that the operations they carry out on small whole numbers work for all numbers. Unfortunately, some teachers, and many textbooks, tend to restrict students' experiences to the use of small whole numbers instead of using a wide range of numbers. In the past, this practice could be understood because of the difficulty of pencil-and-paper computation. However, now that students have access to hand-held calculators, no reason exists to restrict so tightly the range of numbers that they use.

Some suggestions

- Use large numbers in problems. For example, move children on from "three people share 15 apples" to "three people share $42" to "three chickens weigh 1710 grams" and "400 ibises eat 90,000 grasshoppers." Variants of one problem will help children recognize that the same operation applies to each, regardless of the size of the numbers. Use calculators so that students' attention is focused on the operation and not on the numbers.
- Number patterns can help develop ideas of decimals (see fig. 23.4). Too often, pattern work is restricted to whole numbers. Patterns that apply to whole numbers can be investigated to see

whether they work with decimals, fractions, and negative numbers. Sometimes interesting reasons exist why they do not work. One example, of a student's seeing the pattern as "adding a 0" rather than multiplying by 10, is given later in this article. Another interesting example can be seen in the article "Liz's Pattern" (Land and Becher 1997) in the algebraic-thinking focus issue of *Teaching Children Mathematics.*

Use a calculator to find the output numbers when the rule is "Divide by 10."

Input number	Output number
50 000	....
5 000	...
500	..
50	5
5	0.5
0.5	...
0.05	....
0.0005	.....

What do you notice? Predict what happens if the rule is "Divide by 100." Check your answers on a calculator. Predict what happens if you start with a number that is not a multiple of 10; try 98 765, for example. What is the smallest number that you can get on your calculator?

Fig. 23.4. Extending a number pattern to include decimals

- Take opportunities to generalize mathematical ideas beyond whole numbers. See the 1997 focus issue of Teaching Children Mathematics, in which Lubinski and Otto describe how a first-grade class extended beyond whole numbers to halves and quarters the mathematical principles involved in a story about sharing cookies. As they continued to work with the situation of sharing twenty-four cookies among more and more children, the class began to understand the inverse multiplicative relationship linking the two variables—the number of children and the number of cookies, or fraction of a cookie, that each could receive. The teacher's questioning kept the focus on the relationships rather than on just the numbers involved.

Understanding Properties of Numbers

The problem

One of the most frequent algebraic acts is renaming expressions, for example,

$$a - (b - c) = (a - b) + c.$$

To understand what is going on here, students need to know how subtraction works. If they forget the rules for operating with algebraic expressions, they should be able to go back to arithmetic and see what happens in specific instances with numbers instead of letters. Very often, however, students' knowledge of number properties is not strong enough. Children must strengthen their intuitive knowledge of number properties so that they know, for example, "If you subtract 1 less, then the answer will be 1 more." They must be confident, without checking by computing both sides, that

$$3037 - (258 - 1) = (3037 - 258) + 1$$

so that they can later generalize to the algebraic statement

$$a - (b - c) = (a - b) + c.$$

Activities such as those shown in figure 23.5 help to develop this knowledge.

Children who have a weak understanding of number properties are often not concerned about the effects of regrouping or reordering numbers. Even in the middle years, some students are not aware that it matters how numbers are grouped or which operation is done first. For example, fourteen-year-old students working on a problem that required them to work out $3 \times 5 + 9 \div 3$ came up with three different answers. Problems like the one in figure 23.6 can help students understand why grouping and ordering are important and how parentheses are used to show which interpretation is intended.

Describing Patterns and Functions

The problem

In working with middle school students, we have found that they often see relationships in patterns that are valid but cannot be expressed in mathematical symbols. For example, for the following table,

1. Work out this subtraction problem. Use a calculator if you want to.

$$\begin{array}{r} 3037 \\ -\underline{258} \end{array}$$

Next write down the answers to these subtraction problems, without working them out. Explain how you know that the answers are correct.

$$\begin{array}{r} 3037 \\ -\underline{259} \end{array} \qquad \begin{array}{r} 3037 \\ -\underline{257} \end{array} \qquad \begin{array}{r} 3037 \\ -\underline{268} \end{array} \qquad \begin{array}{r} 3037 \\ -\underline{358} \end{array} \qquad \begin{array}{r} 3037 \\ -\underline{158} \end{array} \qquad \begin{array}{r} 3037 \\ -\underline{288} \end{array}$$

2. Work out this problem.

$$\begin{array}{r} 6214 \\ -\underline{1989} \end{array}$$

Write down six other subtraction problems that would also have this answer. Explain how you changed the numbers 6214 and 1989.

– ____ – ____ – ____ – ____ – ____ – ____

Fig. 23.5. Developing confidence in understanding number properties: Taking away more, getting less

1. Carla, Dan, and Ella were working out the answer if you start with 3, multiply by 5, add 9, and divide by 3. They wrote the question down as $3 \times 5 + 9 \div 3$. They got different answers.

   Carla said, "I know that 3×5 is 15, and I know that $9 \div 3$ is 3, so it's 15 plus 3. The answer is 18."

   Dan said, "No, that's wrong. There's 3 at the front and then $\div 3$ at the end. The 3s cancel out, so you've just got 5 plus 9, that's 14."

   Ella said, "I think the answer is 8 because you have to do it in order. Three times 5 is 15, then add 9 and you get 24, and then divide by 3 and you get 8."

   Is Carla right? Is Dan right? Is Ella right?

2. Can you change the answers for any of these calculations by putting quantities in parentheses?

   (i) $6 \times 12 \times 3$ (ii) $6 \times 12 \div 3$ (iii) $6 \times 12 - 3$ (iv) $6 \times 12 + 3$

   Try some other sets of three numbers. What do you notice? What happens if you use fractions? Try

$$6\times\frac{1}{3}\times 3$$

   and

$$6\times\frac{1}{3}+3.$$

Fig. 23.6. Three answers to one problem: Grouping and ordering

x	1	2	3	4	5	6	7	8	...
y	9	19	29	39	49	59	69	79	...

some students noticed the patterns in each row—"x goes up by ones and y goes up by tens." A few students said that "y goes up by 9s," having counted how many numbers were missing in the gap between 9 and 19. Some noticed that y increases at a faster rate than x—"For every x, there are ten y's." Some noticed a link between each x and the y to its lower right—"The x tells you what the tens digit is in the next y." Some found rules linking x and y—"To work out y, put a zero onto x and take 1" or "Take 1 off the x, and write 9 after it." None of these descriptions can be written algebraically, although the students were able to use them to continue the table.

Students need opportunities to talk about the patterns they see and the ways to describe them mathematically. The student who said, "Put a zero onto x" needs to know that the operation involved is "multiply x by 10." The students who thought that the difference between 9 and 19 is 9, because they counted nine missing numbers, need to develop their understanding of "difference between" in terms of addition and subtraction, moving beyond simply counting.

Some suggestions

- Let children become familiar with the many forms in which simple mathematical relationships can be expressed. For example, if $A = 30$ and $B = 15$, we can say that "A is twice B," "B is half of A," "B doubled gives A," "A is 15 more than B," "B is 15 less than A," and "The difference between A and B is 15." Children need to hear all these forms used by their teachers and their classmates, articulate them themselves, and know how one form can be changed to another.

- Encourage students to recognize both the successor and the functional relationship in a sequence, such as 3, 6, 9, 12, 15, 18, 21, ... As well as knowing to "continue adding 3," they need to see the pattern $1 \times 3, 2 \times 3, 3 \times 3, 4 \times 3, \ldots$ and recognize the operation "multiply by 3."

- Encourage students to look for a functional relationship between two variables instead of just looking at changes in one variable. For example, use tables showing input and output numbers that are not arranged sequentially. An example is shown in figure 23.7. See Willoughby (1997) for a gradual approach to the concept of function across the grades.

Input number	2	7	9	5	10	6	20	100
Output number	6	11	13	9	14	_	_	_

What is the rule connecting input and output? Use your rule to work out the missing numbers in the output row. What will be the output number if the input number is 2.5? 0.7? 3999?

Fig. 23.7. "What is the rule?" Specifying a functional relationship as a general rule

Conclusion

Algebra is that part of mathematical language that has been designed to express general relationships among numbers. To learn algebra, students need number knowledge that goes far beyond arithmetic calculations and basic skills. In particular, they must understand general properties both of numbers and of the operations that combine them. The activities described in this article offer teachers ideas for giving their students a flying start to algebra.

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Children's Understanding of Equality: A Foundation for Algebra

by Karen P. Falkner,
Linda Levi, and Thomas P. Carpenter

MANY states and school districts, as well as *Principles and Standards for School Mathematics: Discussion Draft* (NCTM 1998), recommend that algebra be taught in the early childhood years. Although young children often understand much more than traditionally thought, adults can have difficulty conceptualizing what would constitute appropriate algebra for the early childhood years. Fifteen teachers and three university researchers are currently involved in a project to define what algebra instruction can and should be for young children. In this article, we discuss the concept of equality, which is a crucial idea for developing algebraic reasoning in young children.

Misconceptions about the Equals Sign

Even though teachers frequently use the equals sign with their students, it is interesting to explore what children understand about equality and the equals sign. At the start of this project, many teachers asked their students to solve the following problem:

$$8 + 4 = \square + 5$$

At first, this problem looked trivial to many teachers. One sixth-grade teacher, for example, said, "Sure, I will help you out and give this problem to my students, but I have no idea why this will be of interest to you." This teacher found that all twenty-four of her students thought that 12 was the answer that should go in the box. She found this result so interesting that before we had a chance to check back with her, she had the other sixth-grade teachers at her school give this problem to their students. As shown in table 24.1, all 145 sixth-grade students given this problem thought that either 12 or 17 should go in the box.

Why did so many children have trouble with this problem? Clearly, children have a limited understanding of equality and the equals sign if they think that 12 or 17 is the answer that goes in the box. Many young children do, however, understand how to model a situation that involves making things equal. For example, Mary Jo Yttri, a kindergarten teacher, gave her students the problem $4 + 5 = \square + 6$. All the children thought that 9 should go in the box. Yttri then modeled this situation with the children. Together, they made a stack of four cubes, then a stack of five cubes. In another space, they made stacks of nine and six cubes. Yttri asked the children if each arrangement had the same number of cubes. The children knew that the groupings did not have the same number of cubes and were able to tell her which one had more. Several children were able to tell the teacher how they could make both groupings have the same number of cubes. Even after doing this activity, however, the children still thought that 9 should go in the box in the equation.

This incident surprised Yttri and the researchers. We had assumed that kindergarten children would have little experience with the equals sign and would not yet have formed the misconceptions about equality

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TABLE 24.1
Percent of Children Offering Various Solutions to $8 + 4 = \square + 5$

Grade	Answers Given					Number of Children
	7	12	17	12 and 17	Other	
1	0	79	7	0	14	42
1 and 2	6	54	20	0	20	84
2	6	55	10	14	15	174
3	10	60	20	5	5	208
4	7	9	44	30	11	57
5	7	48	45	0	0	42
6	0	84	14	2	0	145

demonstrated by older children. Even kindergarten children, however, appear to have enduring misconceptions about the meaning of the equals sign that are not eliminated with one or two examples or a simple explanation. This incident also illustrates that children as young as kindergarten age may have an appropriate understanding of equality relations involving collections of objects but have difficulty relating this understanding to symbolic representations involving the equals sign. A concerted effort over an extended period of time is required to establish appropriate notions of equality. Teachers should also be concerned about children's conceptions of equality as soon as symbols for representing number operations are introduced. Otherwise, misconceptions about equality can become more firmly entrenched. (See "About the Mathematics" on p. 137.)

As Behr, Erlwanger, and Nichols (1975); Erlwanger and Berlanger (1983); and Anenz-Ludlow and Walgamuth (1998) have documented, children in the elementary grades generally think that the equals sign means that they should carry out the calculation that precedes it and that the number after the equals sign is the answer to the calculation. Elementary school children generally do not see the equals sign as a symbol that expresses the relationship "is the same as."

Not much variety is evident in how the equals sign is typically used in the elementary school. Usually, the equals sign comes at the end of an equation and only one number comes after it. With number sentences, such as $4 + 6 = 10$ or $67 - 10 - 3 = 54$, the children are correct to think of the equals sign as a signal to compute.

First and Second Grades

Karen Falkner is currently teaching a first- and second-grade class. Children typically stay in the class for two years. The remainder of this article shows how the children in this class have progressed in their understanding of equality over the past year-and-a-half.

For some time, solving story problems has been an integral part of mathematics instruction in Falkner's class. Students are regularly asked to write number sentences that show how they solved story problems. Falkner expected her students to be successful, therefore, when she first asked them to solve the number sentence $8 + 4 = \square + 5$. To her surprise, the students answered the problem just as research indicated that they would. Most put 12 in the box, and some extended the sentence by adding = 17. The discussion that followed was interesting. Most said that 12 should go in the box because "eight plus four equals twelve." The following excerpt illustrates the class discussion that took place after students had worked on the problem.

Falkner: Is 8 + 4 the same as 12 + 5?
Anna: No.
Falkner: Then why did you put 12 in the box?
Anna: Because 8 + 4 equals 12. See? [Counting on her fingers] It's 8, 9, 10, 11, 12. [Many children nod their heads in agreement.]
Falkner: Did anyone get another answer?
Adam: It is 7.
Falkner: Why?
Adam: Because you have to have the same amount on each side of the equals sign. That's what the equals sign means.
Falkner: I see. Adam, would you say that again?

[Adam repeats his explanation. Other children, considering Adam a class leader, listen attentively.]

Falkner: [Gesturing at the number sentence on the chalkboard.] So, Adam, you say that the equals sign means that however much something is on one side of the equals sign, the same amount has to be on the other side of the equals sign. [Looking at the rest of the class] What do you think about what Adam said?

Anna: Yes, but it has to be 12, because that is what 8 + 4 equals.

Dan: No, Adam is right. Whatever is one side of the equals sign has to equal what is on the other side: 8 + 4 = 12 and 7 + 5 =12, so 7 goes in the box.

The class wrestled with this problem for some time. The equals sign is a convention, the symbol chosen by mathematicians to represent the notion of equality. Because no logical reason exists that the equals sign does not mean "compute," Falkner thought that it was appropriate to tell the class that she agreed with Adam and Dan. Telling the class what the equals sign meant was not, however, sufficient for many children to be able to adopt the standard use of the sign.

Falkner then chose to develop her students' understanding of the equals sign through discussion of true and false number sentences; this discussion builds on the work of Robert Davis (1964). Falkner presented number sentences, similar to the following, to her students and asked whether the number sentences were true or false.

$4 + 5 = 9$　　$12 - 5 = 9$　　$7 = 3 + 4$
$8 + 2 = 10 + 4$　　$7 + 4 = 15 - 4$　　$8 = 8$

The children's reactions were interesting. All agreed that the first sentence was true and that the second was false. They could prove these assertions by a number of means. They were less sure about the remaining sentences.

Falkner: What about this sentence? 7 = 3 + 4. Is it true or false? [Lots of squirming around, distressed faces, and muttering from the class]

About the Mathematics

Children must understand that equality is a relationship that expresses the idea that two mathematical expressions hold the same value. It is important for children to understand this idea for two reasons. First, children need this understanding to think about relationships expressed by number sentences. For example, the number sentence 7 + 8 = 7 + 7 + 1 expresses a mathematical relationship that is central to arithmetic. When a child says, "I don't remember what 7 plus 8 is, but I do know that 7 plus 7 is 14 and then 1 more would make 15," he or she is explaining a very important relationship that is expressed by that number sentence. Children who understand equality will have a way of representing such arithmetic ideas; thus they will be able to communicate and further reflect on these ideas. A child who has many opportunities to express and reflect on such number sentences as 17 – 9 = 17 – 10 + 1 might be able to use the same mathematical principle to solve more difficult problems, such as 45 – 18, by expressing 45 – 18 = 45 – 20 + 2. This example shows the advantages of integrating the teaching of arithmetic with the teaching of algebra. By doing so, teachers can help children increase their understanding of arithmetic at the same time that they learn algebraic concepts.

A second reason that understanding equality as a relationship is important is that a lack of such understanding is one of the major stumbling blocks for students when they move from arithmetic to algebra (Kieran 1981; Matz 1982). Consider, for example, the equation $4x + 27 = 87$. How do you start solving this equation? Your first step probably involves subtracting 27 from 87. Why may we do so? We may do so because we subtract 27 from *both sides* of the equation. If the equals sign signifies a relationship between two expressions, it makes sense that if two quantities are equal, then 27 less of the first quantity must equal 27 less of the second quantity. What about children who think that the equals sign means that they should do something? What chance do they have of being able to understand the reason that subtracting 27 from both sides of an equation maintains the equality relationship? These students can only try to memorize a series of rules for solving equations. Because such rules are not embedded in understanding, students are highly likely to remember them incorrectly and not be able to apply them flexibly. For these reasons, children must understand that equality is a relationship rather than a signal to do something.

Gretchen: Yes, 3 + 4 does equal 7.
Ned: But the sentence is wrong.
Anna: It's backward.
Falkner: But Adam has told us that the equals sign means that the quantity on each side of it has to be equal. Is that true here?
Anna: Yes, but it's the wrong way.
Falkner: Let's try this. [She models the problem, giving one child seven Unifix cubes in a stack and asking him to stand on one side of her. She gives another child a stack of four Unifix cubes for one hand and a stack of three for the other hand. That child stands on the other side of her.] Now, do these two children have the same number of cubes?
Class: Yes.
Falkner: Does it make any difference which side of me they stand on? [She asks them to change places, which they do.]
Class: No, but . . . .

As you can imagine, the fourth number sentence caused confusion for many children. Some children thought the number sentence was true because 8 + 2 does equal 10. Children who had a firm understanding of equality were able to explain that this number sentence was not true because 8 + 2 is 10 and 10 + 4 is 14 and 10 is not the same as 14.

When Falkner came to the final sentence, 8 = 8, the class was quite disturbed. Anna spoke for the students when she said, "Well, yes, eight equals eight, but you just shouldn't write it that way." In the few remaining weeks of school, Falkner continued to give problems to her students with the equals sign in various locations.

The Next Year

In the fall, Falkner posed the same problem, 8 + 4 = □ + 5, to her class. A few, but not all, of the children who had been in the room the previous spring correctly solved the problem. Many new first graders proudly put 12 in the box; others looked at the sentence in confusion and asked for help. A discussion similar to the one in the spring ensued. This time, however, a few children understood the notion of equality and enthusiastically explained why the number 7 belonged in the box. Lillie gave the most spirited explanation. "The equals sign means that it has to be even. The amount has to be the same on each side of the equals sign. [Gesturing with her hands] It is just like a teeter-totter. It has to be level."

This class discussion was the first of several about similar open number sentences. Each discussion had its doubters, as well as children who once again explained the idea that each side of the equals sign had to "equal" the same amount. As Falkner listened to the discussions, noted who was talking, and looked at facial expressions, it appeared that the children were beginning to grasp this notion of equality but that the concept was not easily or quickly understood. Falkner was convinced that the notion of equality would take time for all the children to understand, and she returned to it often as the year progressed.

Falkner integrated discussion of equality throughout the school year in two ways. First, she continued to present open number sentences in which she varied the location of the unknown. Some examples of these open number sentences included the following: □ = 9 + 5, 7 + 8 = □ + 10, and 7 + □ = 6 + 4. Second, she presented true and false number sentences, such as those in the examples, to encourage children to reflect on the meaning of the equals sign. She also had the children write their own true and false number sentences. The tasks that Falkner used to build children's understanding of equality were also tasks that build their understanding of number operations.

As the year progressed, more and more children began to understand equality. In March, the class had the following discussion:

Falkner: Look at this number sentence: 8 + 9 = □ + 10. What should go in the box?
Carrie: It should be 17.
Skip: But 8 + 9 would equal 17, so 17 + 10 would equal 27, so 17 isn't OK to put in the box.
Myra: Right; 17 + 10 does not equal 17.
Ned: I think that 7 goes in the box; 7 + 10 is 17 and 8 + 9 is 17. Both sides are even. [The class generally agrees, although Carrie is not yet convinced.]
Faulkner: Think about what we know about the equals sign. Look at this number sentence: 4898 + 3 = 4897 + □. Can you figure this one out without even doing the addition?
Larry: I think that 4 goes in the box; 4897 is 1 down from 4898, so you need to add 1 more to 3.
Falkner: Did anyone do it a different way? [Children shake their heads. In general, the class agrees that Larry's way gives the right answer and is easy.]

Such discussions about number sentences gave the children an important context for discussing equality throughout the school year. As the year progressed, discussions about equality became integrated with discussions about other algebraic arithmetic concepts. In the following example, the children discuss a much more sophisticated problem that involves an understanding of variables and operations, as well as equality.

Falkner asked the class to look at the sentence $a = b + 2$. She said that the sentence was true and asked the class which was larger, *a* or *b*? Children who think of the equals sign as a signal to do something would have trouble with this problem. Because 2 is added to *b* and nothing is added to *a*, they might think that *b* is larger. The class first agreed that *a* and *b* were symbols for variables, just as a box or triangle were. The class then quickly agreed that *a* was larger, and their arguments for that position clearly indicate a sophisticated understanding of equality.

Falkner:	Why do you think that *a* is larger?
Anna:	They split the *b* and 2 apart; *a* brings them together.
Jerry:	I think *a* [is larger]. That plus 2 is part of *a*.
Myra:	Yes, *a* has to be bigger because whatever $b + 2$ is has to be higher than *b*, because you combine them.
Anna:	Right; *a* has the + 2 in it and *b* doesn't.
Lillie:	Together they have to be the same; $b + 2$ has to be the same as *a*.

Conclusion

Discussions such as these, which involved an ever-growing number of children, indicate that the children have learned to see the equals sign as a symbol describing a relationship rather than as a "do it" sign. Because this article was written before the end of the school year, we have not collected summary data on children's understanding of the problem $8 + 4 = n + 5$ in this class. In a pilot study involving a similar first- and second-grade classroom in the same town, however, we found that at the end of the year, fourteen out of sixteen children correctly answered that 7 should go in the box.

As we reflect on our introduction of the notion of equality and the equals sign to this class and others, we continue to be amazed at the interest and excitement that the children bring to the discussions. Lillie uses her teeter-totter metaphor with the enthusiasm of a child ready to play on one. Skip is genuinely outraged that anyone should fill in a blank so that an equation reads $17 = 27$. These are not the bored comments of children looking forward to recess but the excited contributions of children who are exploring a new world of thinking and communicating mathematically and who are enjoying the power of that new knowledge. These children are developing an understanding of equality as they learn about numbers and operations. This understanding will allow them to reflect on equations and will lay a firm foundation for later learning of algebra.

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What Comes Next? The Mathematics of Pattern in Kindergarten

by Karen Economopoulos

PATTERNING activities are a staple of the kindergarten curriculum. If you ask any kindergarten teacher, she or he is likely to consider the study of patterns to be an essential part of the mathematics program. It is common to walk into an early childhood classroom and see students making repeating patterns with a variety of materials. Long trains of interlocking cubes might be snaking their way across the floor, or pattern "walls" made from pattern blocks might be stretching across a table. In the block area, one might see an alternating design made from triangular and rectangular prism-shaped blocks adorning the wall of a block structure. On the bulletin boards may be seen evidence of children's work with patterns in the form of colorful green-yellow-green-yellow handprints or pattern "snakes" made from paper pattern-block shapes glued onto thin strips of paper. In all these instances, children constructed and extended a design that repeated in a regular and predictable way. They were making patterns.

What, though, do young students understand as they work with patterns? What is the significant mathematics in these activities for five- and six-year-olds, and how do these ideas and activities connect to subsequent work with mathematics? The NCTM's Standards Addenda book *Patterns* (Coburn et al. 1993) states that in kindergarten "children should focus on regularity and repetition in motion, color, sound, position, and quantity [and] be involved in recognizing, describing, extending, transferring, translating, and creating patterns" (p. 2). Furthermore, to understand the concept of pattern, children must recognize the predictability and repetition that patterns imply in the following forms (Coburn et al. 1993, 2):

- Some patterns repeat the basic unit: red-red-blue, red-red-blue.
- Other patterns seem to grow: clap-hop, clap-clap-hop, clap-clap-clap-hop.
- Patterns may emerge in unexpected ways: when we explore with manipulatives, we often discover patterns.
- Patterns are part of the world we have built: the green light follows the red light, and the traffic begins to move.

As a way to think about these questions and ideas about patterns, as well as other mathematical topics in the early elementary curriculum, a group of kindergarten teachers met regularly during the school year to discuss their students' thinking about mathematics. Promoting discussion among teachers about how children learn mathematics has been shown to be an effective way to encourage teachers to think more critically about mathematics teaching (Simon and Schifter 1991; Wilson and Ball 1991). The Kindergarten Teachers Group was modeled on this premise and found that doing mathematics together helped them learn more about the mathematics they teach.

During the meeting times devoted to the discussion of patterns, the kindergarten teachers investigated mathematics problems involving patterns and reflected on classroom vignettes that they had recorded when their own students were engaged in patterning activities. Like most kindergarten teachers, they had viewed patterning activities as an accepted and straightforward part of their curriculum for many

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The Kindergarten Teachers Group, referred to in this article, is part of Where's the Math in the Kindergarten Classroom?, a staff development project of the Education Research Collaborative at TERC, Cambridge, Massachusetts. The project is funded by the Exxon Education Foundation.

years. As they considered their work with patterns together, they began to examine more carefully the activities that they had taken for granted and to think together about the mathematical ideas underlying these activities. One teacher made this comment:

> I've been doing pattern activities with my students for years. Most kindergarten teachers do lots of patterning stuff with their kids. But why? Why do we spend all this time on pattern—sometimes too much time doing the same type of thing? Like the kids build red-blue-red-blue patterns, and then we have them do it with lots of different materials; but once they understand red-blue-red-blue, what is it that they should be doing next? What is it that I [the teacher] should be doing in terms of questioning or providing the next activity that will deepen their understanding of pattern and stretch their thinking to a new level?

As the teachers described the types of patterning activities that they offered their students, it became clear that most of the tasks were focused on children's copying an existing pattern, creating their own repeating pattern, or extending a pattern that had been created by themselves or someone else. In all examples, the teachers thought that the purpose of these activities was to engage students in focusing on "What comes next in this pattern?"

For many kindergartners, simply deciding what comes next is not simple or obvious. To copy, create, and extend a pattern, students must examine the pattern carefully. They must look for relationships among pattern elements and explore how that information can be used to determine what to add to the pattern. As they begin to see relationships, they begin to see that deciding what comes next in a pattern is not a random choice. When students first encounter patterns, however, they may be seeing only that red comes after blue and blue comes after red. At some point they must make the connection that red-blue is a fundamental unit of the pattern. Patterns may have elements that alternate, repeat, increase, or decrease in a regular way. Being able to describe these regular relationships leads to predicting how the pattern will continue. For young students, recognizing that patterns are predictable is an important mathematical idea.

Should helping students understand that patterns are predictable be the focus of our work in kindergarten? Certainly, this idea is important, but what are the next steps in extending this understanding or using it in a different way? Being able to generalize about a pattern and use known information to predict unknown information is truly the powerful aspect of patterns. To generalize and predict, students must move from looking at a pattern as a sequence of "what comes next" to analyzing the structure of the pattern, that is, seeing that it is made of repeating units.

As a way of extending the teachers' conversation about pattern, they were asked to consider the following two activities that they might try with their students.

"What Comes Here?"

"What comes here?" is an activity in which students use information that they already know about a pattern to predict unknown information. In this activity students build a pattern using color tiles or interlocking cubes. They cover the last four or five elements in their pattern with small paper cups. (See fig. 25.1.) Students take turns pointing to a cup and guessing the color of the tile underneath. In this game, the focus of the activity shifts from what comes next, although this question would be possible, to what comes here as they use known information to predict unknown information. For example, a student's pattern might show green-blue- green-blue-green tiles, with the next four tiles each covered by a cup. A student might point to the fourth cup, and another student would try to decide the color of the tile under the cup.

As teachers played this game, they predicted how their students might decide on the hidden colors. All agreed that a range of responses would be received, from random guesses to predictions based on the given information. Teachers were very interested in observing how students would determine the hidden color. Would they need to begin with the first cube in the sequence and "read" the pattern in order from left to right, or would they do what many of them did and say, "I looked at the last color, and that was green; so in my mind I just went green-blue-green and knew that the next one had to be blue." Although these two strategies seem similar, the first one may suggest that students are focusing on "what comes after what," whereas in the second, students seem to be moving

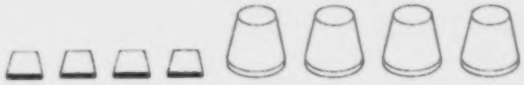

Fig. 25.1. What comes here?

toward thinking about green-blue as the smallest repeating unit in the pattern.

The teachers became interested in not only whether students could do this activity but also how they would approach the task. They realized that they would know how their students were thinking about the task only if in addition to their observations, they asked students to explain their thinking. For these teachers this activity was not only something new to try but also an introduction to a new idea in patterning—the importance of identifying the repeating unit in a pattern and using that unit to predict what comes here.

Pattern Towers—Same or Different?

In addition to the "What comes here?" game, teachers were asked to consider how a tower built from alternating green-orange-green-orange cubes was similar to a tower built from blue-yellow-blue-yellow cubes. (See fig. 25.2.) They quickly saw that even though the towers were composed of different colors, their basic composition was the same—that both towers were made up of repeating units with the same structure.

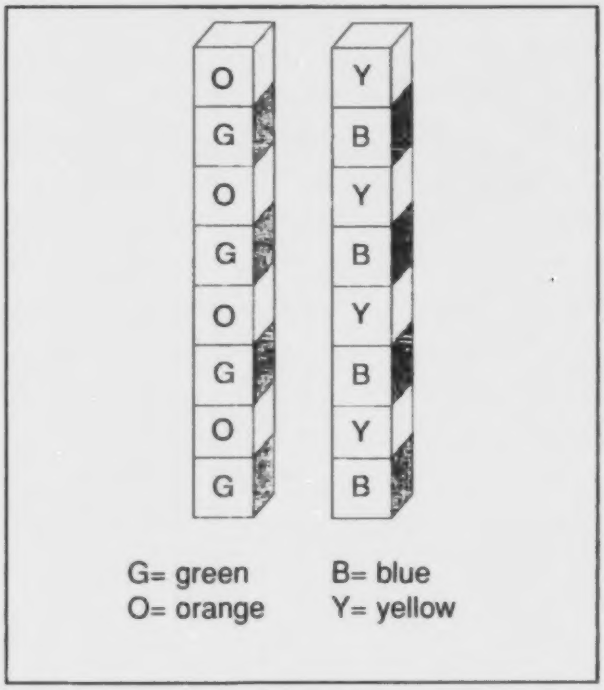

Fig. 25.2. Pattern towers

The discussion turned to the classroom environment, with the teachers predicting what their students might say when presented with the same task of comparing two pattern towers. Many teachers predicted that for young students, color would be an overwhelming attribute that would make these towers seem very different, so that blue-yellow would not in any way be considered the same as green-orange. Although this prediction was prevalent, most every teacher also recognized the thinking that could be developed with continued exposure to such an activity. They realized that through these kinds of comparison activities, children would learn to be careful observers of the structure that characterizes a pattern.

The teachers' curiosity about this activity seemed to stem from their own interest in thinking about these simple pattern towers in a new and different way. As one teacher commented, "I've been teaching pattern for ten years, and I wouldn't have thought [about pattern] in this way—seeing it in units." For these teachers, the experience of exploring a mathematical idea for themselves illuminated some important mathematics for them, which in turn led to questions about their work with children and enthusiasm about exploring some new ideas with their students. This experience emphasized to us all the importance of exploring mathematics ourselves as a way to make decisions about teaching.

In both of these activities, students work with the idea that a pattern is predictable and that it is made up of units that repeat. By looking at the unit of a repeating pattern, one is able to make generalizations that can help in predicting unknown information. To do this sort of work, it became clearer to teachers that although students need a great deal of varied experience with copying and extending patterns and with creating their own patterns, they also need opportunities to describe their patterns to others and to consider how patterns are the same and different.

Seeing Pattern as Structure: Connection to Later Work in Mathematics

Although we should not expect early childhood students to understand fully the complexities of pattern, we can engage them in activities that involve thinking about the predictability and the structure of patterns—that patterns are composed of units that repeat. Young students' work with these patterning ideas will connect significantly to their later work in mathematics.

Because our number system is built on a system of patterns and predictability, students must be able not only to identify the patterns that they see but also to give reasons and evidence for why the patterns exist. For example, as students begin to examine closely the sequence of numbers, they soon recognize that many patterns and relationships are evident. When students count by fives, they quickly notice a pattern in the counting numbers of 5, 10, 15, 20, 25, 30, 35, and so on, and that in this counting sequence, the ones digit alternates 5, 0, 5, 0. As they examine this pattern more closely, they can explain that it is not just coincidental but is in fact linked to the base-ten structure of the number system and that every group of five that is added makes either half a group of 10 (5, 15, 25) or a whole group of 10 (10, 20, 30). Students may then generalize that any number that is a multiple of 5 will end in either a 0 or a 5. From this knowledge they can then predict the types of answers that they will get when they multiply a number by 5.

Most early primary teachers may never encounter students using their knowledge of pattern in these ways. In fact, understanding and explaining the pattern of counting by fives may seem quite far away from pattern trains and predicting "What comes here?" However, for students to use patterns in powerful ways, they must build an understanding of what patterns are, beyond finding what comes next. Likewise, although the work of kindergarten may seem to be clear and simple, when one group of kindergarten teachers engaged in mathematics together and considered the work of their students, they began to think more deeply about an area of their curriculum that they had always taken for granted. They returned to their classrooms with new insights about a once familiar topic—insights that allowed them to further their students' exploration of ideas about pattern.

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ACTIVITIES

Number and Operations; Algebra

Isaacs and Carroll: "Strategies for Basic-Fact Instruction"

One of the goals for the Number and Operations Standard states that instructional programs should enable all students to—

> *understand numbers, ways of representing numbers, relationships among numbers, and number systems.*

1. Describe how the use of the hundred-chart (fig. 21.1) and the ten-frames (fig. 21.2) promote this goal.
2. Show two ways to use the hundred-chart to illustrate 12 + 13.
3. Describe all the information that can be derived from using ten-frames to find 7 + 6.
4. How do you think that using the hundred-chart and ten-frames can promote a child's mastery of basic facts? Use your answers to the previous two questions in your explanation.

Curcio and Schwartz: "There Are No Algorithms for Teaching Algorithms"

Figure 22.1 presents three subtraction algorithms (a), (b), and (c). For each algorithm, do the following:

1. Describe the sequence of procedural steps needed to carry out each subtraction.
2. Explain mathematically why the steps work.
3. Use each algorithm to calculate 453 – 178.
4. Which method do you prefer, and why?

Section 1: Number and Operations Standard

Write a reaction paper covering the three articles in this section.

1. Discuss how these articles do or do not exemplify the goals listed for the Number and Operations Standard.
2. Describe three pieces of information, insights, or deeper understandings that you gained from reading the articles. Alternatively, describe three things with which you disagreed.
3. Select an example of a student's explanation from one of the articles, and describe your reaction to this particular episode.

MacGregor and Stacey: "A Flying Start to Algebra"

1. Answer the questions posed in figure 23.5. Be sure to include the requested explanations.
2. Read the problem posed in figure 23.6, and answer all the questions.
3. Find the answers to the questions posed in figure 23.7.
4. Select the activity given in one of figures 23.5, 23.6, or 23.7, and clearly explain how this particular activity addresses one of the "problems" described in the article.

Falkner, Levi, and Carpenter: "Children's Understanding of Equality: A Foundation for Algebra"

Economopoulos: "The Mathematics of Pattern in Kindergarten"

Group Discussion

Many mathematical entities exhibit a process-object duality. As a *process*, such entities are perceived as "something to be done." For example, the expression "4 + 5 " can be regarded as directions to add. As an *object*, the entity is regarded as a thing in itself, having properties. For example, "4 + 5 " can also be regarded as an expression for the number 9, one of whose properties is that it can be decomposed into the sum of the two numbers 4 and 5.

The equals sign and the notion of patterning both exhibit a process-object duality.

1. Describe how each of those entities is used as a process. Give examples.
2. Describe how each can be regarded as an object. List several properties of each object.
3. Identify another mathematical entity that exhibits a process-object duality. Describe how it is used as a process, and list several properties of its objectified form.

For more information on the process-object duality see "The Gains and Pitfalls of Reification —the Case of Algebra" by Anna Sfard and Liora Linchevski (*Educational Studies in Mathematics* 26: 191–228, 1994).

2.

GEOMETRY; MEASUREMENT

GEOMETRY and measurement go hand in hand. The concepts of length, area, and volume are used to identify attributes of shapes, and measurement is used to quantify these attributes. As the following articles illustrate, the development of these concept strands occurs gradually from many rich explorations at the elementary level.

Geometry

INSTRUCTIONAL programs from prekindergarten through grade 12 should enable all students to —

- *analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;*
- *specify locations and describe spatial relationships using coordinate geometry and other representational systems;*
- *apply transformations and use symmetry to analyze mathematical situations;*
- *use visualization, spatial reasoning, and geometric modeling to solve problems.*

Wheatley and Reynolds discuss the importance of mental imagery in facilitating mathematical thought in numerical and geometric settings. The authors present a series of short "image maker" activities that encourage the formation and transformation of mental images. Using these activities helps students develop geometric concepts and vocabulary, as well as self-confidence and a positive attitude toward mathematics.

The article by Fox presents a set of tasks that can be used to assess elementary school students' levels of two-dimensional geometric reasoning. These levels represent a sequential development of geometric understanding and consist of reasoning by resemblance, reasoning by attributes, and reasoning by properties. The results of these open-ended tasks can be used to create a match between instruction and students' existing reasoning levels.

Cockroft and Marshall emphasize the importance of precise thinking in developing an understanding of geometric properties in two- and three-dimensional space. The authors illustrate their position with an entertaining vignette in which two children discuss the difference in meaning when a shape is called a "thick circle" or a "thin cylinder."

The fourth article examines the role of definitions in developing geometric understanding. Keiser describes a sequence of activities in which the concept of angle was gradually constructed through reflective encounters within a variety of contexts. Examples of students' journal entries illustrate the evolving nature of the students' concept image of angle and emphasize the complexity of this geometric notion.

"Image Maker": Developing Spatial Sense

by Grayson H. Wheatley and Anne M. Reynolds

JUST as it is useful to have a mental map of streets in a city, it is also useful to have mental images of mathematical patterns and relationships. A student who has constructed a network of image-based mathematical meanings can more easily devise solutions to problems. If a student knows only one way and if the task is not straightforward, he or she may not be able to solve the problem. The ability to construct and transform mental images leads to flexibility and power. In doing mathematics, it is advantageous to know more than one way to solve a problem or complete a routine task. Mental images play a crucial role in all aspects of mathematics.

Developing spatial sense, as well as number sense, as described in NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989), is a central goal of mathematics instruction that engenders problem solving in particular and doing mathematics in general. A strong spatial sense allows students to formulate image-based solutions to mathematics problems. In geometry, having a mental image of a parallelogram is fundamental. Without spatial sense, a student may only act mechanically with shapes and symbols that have little meaning.

Many teachers are shifting from a procedural to a conceptually oriented mathematics curriculum. Rather than emphasize memorization and computation, teachers are focusing on concepts and problem solving. Given the demands of an information age, the abilities to conceptualize and to create assume greater importance. Curricula that dwell on memorizing formulas and worked-out examples no longer serve our students well. In this shift from procedures to concepts, spatial sense gains increased importance.

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The activities described in this article are designed to develop mental imagery and the ability to transform these images. Using images facilitates mathematical thought in numerical and geometric settings.

"Image Maker"

"Image maker" is designed to encourage the formation and transformation of mental images. It is a whole-class activity and may be used as a class opener or as a full lesson. The teacher shows a line drawing for three seconds on an overhead projector and asks students to draw it on their papers. Drawings that can be interpreted in more than one way promote especially rich discussion. As students describe what they saw, other students may come to see the figure differently. Because students must draw from a constructed mental image, the activity is quite different from copying a drawing that is in full view. "Image maker" engenders the construction of geometric shapes, which in turn enhances the learning of geometric concepts. A source of figures can be found in Wheatley (1996).

A second brief exposure follows the students' initial drawing attempt. Students are asked to "Draw what you saw." The past tense is used because the students must work from the mental images they construct—the figure is not observable while they are drawing. A discussion of their interpretations and methods of drawing is an integral part of the "image maker" activity. A typical whole-class discussion for figure 26.1 follows.

T: What did you see?
Katrina: I saw a hexagon with an X in it.
Rosie: I saw two diamonds and two triangles.

T: Please explain. Come to the overhead projector, and point out those shapes you described.
Robert: Oh, yeah! I see what she means.
Dorie: It looks like an envelope opened up.
T: What do you mean? Does anyone understand what Dorie is saying?
Pat: I see it now [goes to the overhead projector]. These are the flaps. If they were folded down, it would be an envelope.
Others: Oh, I see it now.
Juan: It looks sorta like a cube, but a line is missing.
T: I don't see that. Come up here and show us how you are thinking.
Juan: See, if there was a line right here, it would be a cube.
Christy: Yes! I see the cube.
T: Isn't it interesting? I showed you a geometric figure, and you came up with many different ways of seeing it. I am impressed with your thinking. We will do some more of these another day.

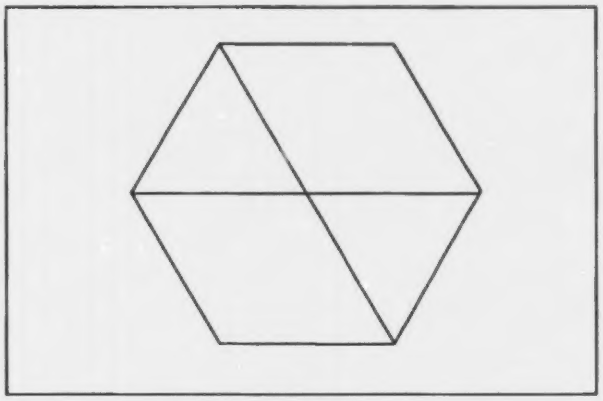

Fig. 26.1. "Image maker" 1

A second "image maker" shape is shown in figure 26.2. When this figure was shown, fourth grader Danielle first drew a hexagon and then three interior segments. It was clear from her action and subsequent comments that she had constructed a vivid two-dimensional image of the figure, and she drew it with confidence. This drawing had previously been used with many children and adults, and most of them reported seeing a cube. A variety of interpretations result when "image maker" shapes are used.

When a different drawing (see fig. 26.3) was used with third-grade students, their responses included the following:

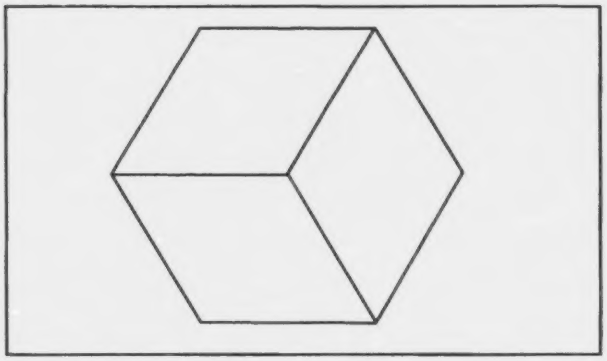

Fig. 26.2. "Image maker" 2

- "It looks like a room."
- "I see a hallway."
- "It looks like a pyramid with the top cut off."
- "I saw a small square connected to a big square."
- "I see a skylight."

Each individual gives meaning to his or her experiences in his or her own way. Some students will see figure 26.3 as a two-dimensional figure; others will give it a three-dimensional interpretation. Individuals respond meaningfully to this task because they are free to interpret the drawing in any way that makes sense to them. When students draw, they work from a re-presented—created from memory—image because the figure is not in view. Finally, they are asked to describe what they saw and explain how they drew their sketch. As students listen to the ways in which others saw the figure, they are stimulated to reflect on their constructive activity and consider other interpretations. Often, students will describe new ways of viewing the figure when listening to the descriptions of others.

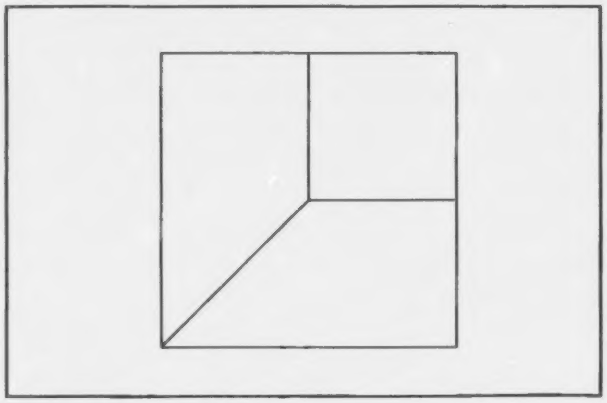

Fig. 26.3. "Image maker" 3

Although individual learners construct their own mathematics, this activity is greatly influenced by interactions with others. Classroom discussions of "image maker" shapes in which students explain their thinking rather than parrot back something that has been explained to them are extremely important in the learning process.

Some children and even some adults may at times have difficulty making a drawing that satisfies them. Note that the manner in which this activity unfolds gives each student a chance to be successful. Some will be satisfied with their drawing after the first exposure. Others will not have finished their drawing until after the second look, and still others will draw it only after the image is exposed for discussion. Each person has the opportunity to complete a drawing without his or her efforts being judged by others.

In many situations mentally transforming an image is important. For example, imaging comes into play when students look at different-shaped boxes on a shelf and plan how to position them in a shipping carton or when they look at newspaper clippings and plan how to position them in an album. The solution to many mathematics problems depends on imagining various positions of lines in a figure. Regular experiences with constructing images help students devise solutions to problems that they see for the first time.

One goal of "image maker" is to help students understand that problems can be solved in various ways and that they can use methods that make sense to them instead of depending on a teacher to demonstrate the way. Because students can give meaning to a figure in various ways, all of which are "correct," they come to realize that they can devise their own methods of doing mathematics without waiting to be shown a particular procedure; they learn to learn.

Teachers report the following benefits of "image maker."

"Image maker" develops geometric concepts and vocabulary. Many geometric shapes, such as parallelograms, triangles, rectangles, and trapezoids, appear in "image maker" figures. In describing the shapes that they see and their drawings, students use such words as *box, diamond,* and *oval.* Teachers can use this opportunity to introduce standard names. Students then associate a geometric term with a shape that they have mentally constructed in the "image maker" activity.

"Image maker" promotes imaging as a brain activity. If a part of the brain is not used, it becomes less functional. "Image maker" encourages the development of the regions in the brain that process, form, and transform images.

"Image maker" creates a mind set that is conducive to meaningful learning. Learning is an active process of sense making. In "image maker," students give meaning to their visual experiences, with the result that many students develop the intention of making sense, which then influences all their mathematical experiences.

"Image maker" builds a basis for number development. We have found that "image maker" carries over to arithmetic. As students develop the intention of making sense, they construct number patterns more easily and give more meaning to performing arithmetical operations—computation becomes less mechanical.

"Image maker" encourages students to make sense of their activity. Too often in school mathematics, students learn procedures that have little meaning for them. Some students may seem to have little promise as mathematics students in a skills-based program, but they may come alive in "image maker" activities and realize they that can think for themselves and succeed in mathematics.

"Image maker" gives teachers important information about students' ways of knowing. As students engage in "image maker" tasks, the teacher comes to know much about their reasoning and capabilities. Students who appear to be good mathematics students when assessed on the basis of their computation skills may not excel at "image maker" pursuits, whereas some students who may have had little success at computing come alive with "image maker."

"Image maker" helps students develop self-confidence. In "image maker" activities, students have successes that help them believe in themselves. This confidence carries over into other activities.

"Image maker" helps students develop a more positive attitude toward mathematics. Because students enjoy "image maker" so much, they develop a more positive attitude toward mathematics and are more ready to engage in other mathematics activities.

"Paper Folding"

Another activity that has proved effective in promoting imagery is "paper folding." In this activity, the teacher folds a piece of paper in half once or twice, cuts off parts, and asks students to draw what the paper will look like when unfolded (see figs. 26.4 and 26.5). Whereas "image maker" promotes the formation and re-presentation of images, "paper folding" affords opportunities for students to develop dynamic imagery. In drawing the shape of the paper while looking at the still-folded paper, students can transform the image and "look at it in their mind." Many mathematics tasks are accomplished by imaging a transformation of a geometric shape. For example, word problems can often be solved by drawing a diagram. To draw the diagram, the individual must first form a mental image of the appropriate diagram.

In "paper folding," the teacher makes cuts in a folded sheet of paper while students watch. The students draw what the paper will look like when unfolded. The materials needed are unlined paper for students and teacher, pencils, and one pair of scissors.

The teacher distributes unlined paper to students and asks them to draw lines vertically and horizontally to partition it into four equivalent regions. Students then have defined spaces to draw four different shapes. The teacher next folds an 8 1/2-by-11-inch blank sheet of paper to form a 5 1/2-by-8 1/2-inch rectangle: "I am folding this piece of paper. Notice that the fold is here [holding hand vertically on the left edge of the folded paper]. Now watch. I will cut this folded paper."

With students watching, the teacher makes the cuts shown in figure 26.4. "Now draw what this paper would look like when unfolded. Remember, the fold is on this side." The folded paper, which has now been cut, is held high so that all can see until students have had an opportunity to draw a shape on their paper. The teacher unfolds the paper and asks students to compare their drawing with the unfolded paper. The teacher then asks, "How did you draw the shape?"

Mike:	I unfolded it in my mind.
Sue:	I drew the shape I saw and then drew the reflection of it on this side. That gave me the solution.

Another "paper folding" example is shown in figure 26.5.

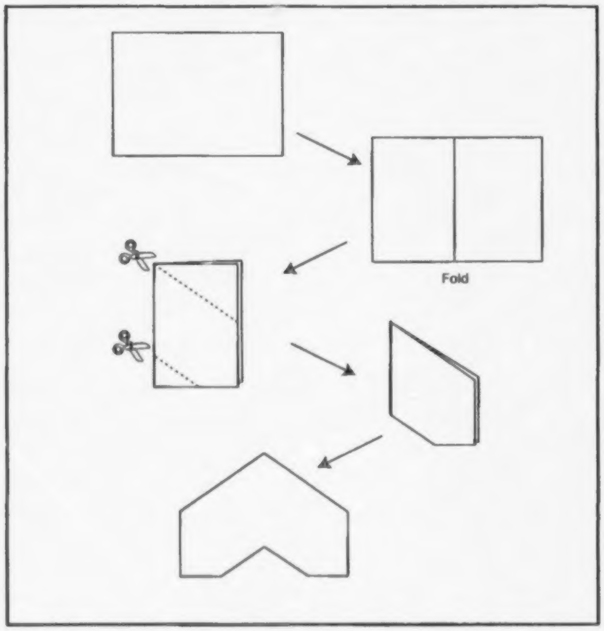

Fig. 26.4. Single paper fold

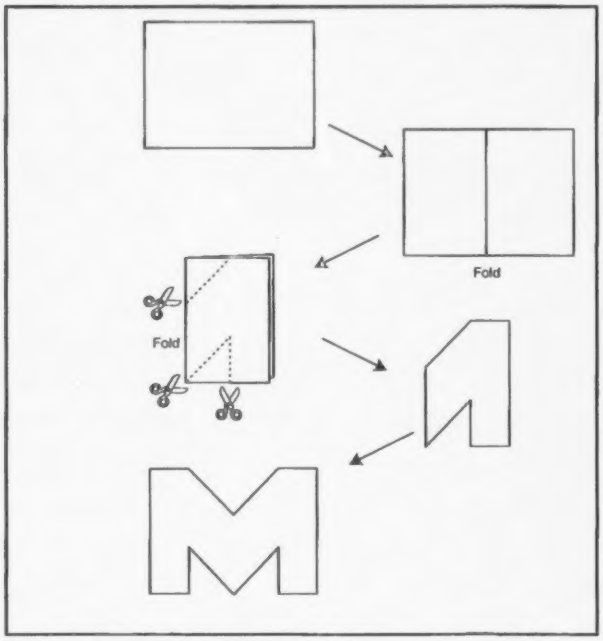

Fig. 26.5. Paper-fold example

These tasks require little preparation or materials but pay big dividends. The initial cuts can be simple and progress to very challenging. Corners can be cut off; and triangles, squares, or rectangles can be cut out of different edges. Students soon see that it makes a big difference whether the cuts are on the fold or on a single edge. After students have had experience with single folds, the double fold presents a new challenge (see fig. 26.6). In this variation, the students must do two mental transformations, that is, unfold twice.

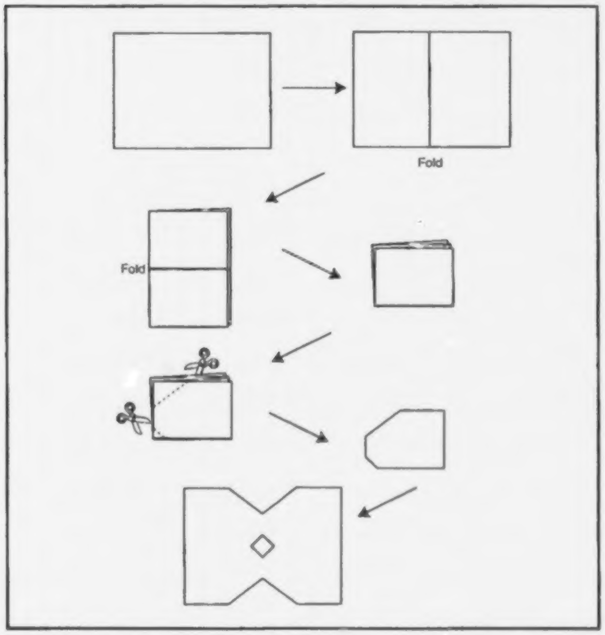

Fig. 26.6. Double paper fold

Both "image maker" and "paper folding" make excellent class openers at any grade level. Teachers have reported that starting a mathematics lesson with these activities helps students get in a frame of mind for doing mathematics. An excellent software program that also helps students learn to enlarge, rotate, reflect, and, in general, develop dynamic imagery of geometric figures is Shape Makers (Battista 1998).

Summary

Students can give greater meaning to their mathematical activities if they are encouraged to construct and transform mental images. In this article, we have discussed two activities that encourage students to use their power to image. Previous studies have shown that students who use imagery are more likely to give meaning to their experiences and be more successful in mathematics, whether it be in a geometric or numerical setting. It is essential that the school mathematics curriculum provide many and varied opportunities for students to construct, transform, and use mental imagery.

References

Battista, Michael. Shape Makers: Developing Geometric Reasoning. Berkeley, Calif.: Key Curriculum Press, 1998. Software.

National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: NCTM, 1989.

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Implications of Research on Children's Understanding of Geometry

by Thomas B. Fox

In my mathematics methods course for preservice teachers, I ask my students to assess a group of elementary school students concerning their level of geometric reasoning. To do so, they use tasks that focus on assessing and extending students' geometric understandings. These open-ended tasks, along with a framework developed from research findings involving children's geometric reasoning, are described in this article. An important aspect of these tasks is that they focus on how students communicate their reasoning so that my preservice teachers can make more informed instructional decisions when planning a follow-up geometry lesson. Research on geometric reasoning has shown that a match between students' reasoning level and instructional tasks is crucial if meaningful learning is to occur (Crowley 1987).

Recent Research on Geometric Reasoning

Much of the recent research on geometric reasoning has focused on the van Hiele model (Crowley 1987). As part of an ongoing research program in Cognitively Guided Instruction in Geometry, Lehrer and his colleagues (1994) have extended this knowledge base by refining and expanding the earlier levels of geometric reasoning commonly exhibited by primary school students, namely, visualization, analysis, and informal deduction. My preservice teachers examined the results from these tasks using this framework.

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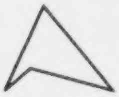
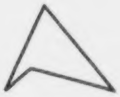
Geometric reasoning by resemblance

The first level at which children reason about shape is termed resemblance. Students at this level classify objects on the basis of their resemblance to other shapes and often rely on irrelevant characteristics of the object. Reasoning by resemblance can be classified as either direct or indirect. Using direct resemblance, students may classify a chevron



as a triangle, because it resembles a "pushed in" triangle. They may also rely on visual stereotypes, as commonly found in textbooks. For example, a "thin" rectangle may not be classified as a rectangle because it is "too thin" relative to the student's rectangular prototype of a door. With indirect resemblance, students may mentally modify a figure to represent another figure that is better known (Lehrer et al. 1994). For example, a chevron



could be considered a triangle by "pulling back" two of its sides until it looks like a triangle.

Geometric reasoning by attributes

As their reasoning matures, students begin to consider a shape's particular attributes or parts, perhaps by using informal language. However, students may not understand the relationships between these attributes. For example, students who do not understand the relationship between the angles and the number of sides of a figure would have to count its angles even though they know the number of its sides. Students who understand such relationships are reasoning at higher levels (Lehrer et al. 1994). At this level, students understand that the chevron is not a triangle but a quadrilateral because it has four sides.

Geometric reasoning by properties

At the next level of reasoning, students analyze a figure in terms of its properties and see relationships between these properties. For example, students at this level know that a square is also a parallelogram because it is a quadrilateral with both pairs of opposite sides being parallel. They also understand that changing a "critical" property of a given shape, such as the length of one side, changes the shape's classification and that changing a "noncritical attribute," such as its size, does not (Lehrer et al. 1994).

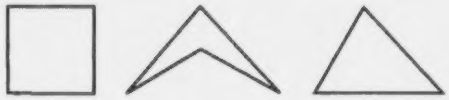
Assessment Tasks

My preservice teachers assess their students' levels of geometric reasoning using the following tasks. I asked the preservice teachers to focus especially on developing questions that encourage students to explain their reasoning. For several tasks, I have included examples of their students' thinking. This thinking has been analyzed using the framework described in this article, and follow-up tasks based on this thinking are proposed.

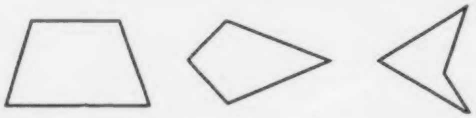
These tasks are open-ended and can be conducted in small groups or in a whole-group setting. Students should be asked to record their thinking on paper to further their own understanding and to help their teacher assess it.

The Polygon Comparison task

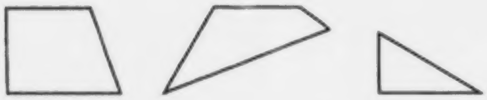
Task description. Students compare and contrast a trio of polygons (adapted from Lehrer et al. [1993]):



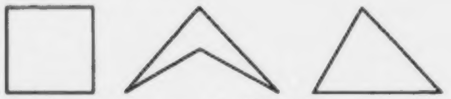
They are also asked to state or write which two figures are most alike and why. This task helps determine whether a student is focusing on a figure's appearance or its parts and properties. Lehrer and others (1993) also suggest presenting this task using these trios:



and



Examples of student's thinking. When asked to compare these three objects,



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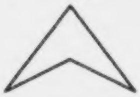
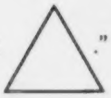
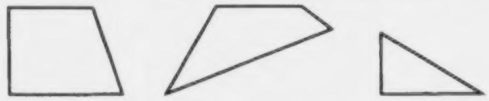
and to describe any similarities and differences among them, one second-grade student reasoned that the two figures that were most alike were the chevron and the triangle, explaining that "if you took the



and drew a line across it, it would become a  ."

Such thinking is characteristic of indirect resemblance. Other students also believed that the second two figures in the trio were most alike. However, the reasoning involved direct resemblance to known geometric figures: "They are both triangles." For these students, appropriate follow-up tasks would include the Polygon Sort activity, which is described subsequently. This activity, especially when completed in a group setting, can encourage students to focus on a figure's attributes, thus attaining the next higher level of reasoning.

When examining a different trio of polygons,



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a pair of fourth-grade students studied the polygons' attributes. They identified the first two figures as having four sides, which indicated that these students were using attributes of the polygons as the primary distinguishing characteristic. Several other students reasoned in a similar manner. For these students, tasks that helped them focus more explicitly on polygons' parts and properties would be especially appropriate. One such task would be the Name and Property Cards task to be described subsequently.

The Polygon Sort task

Task description. Students are asked to sort a set of cardboard or paper polygonal cutouts in a meaningful way and give a rationale for their sort (Fuys, Geddes, and Tischler 1998). If students are unsure of how to proceed, they should be asked how the various polygonal cutouts are different and to sort the figures accordingly. This task can help reveal whether students' reasoning involves appearance, attributes, or properties.

To best determine the level of students' reasoning about various subclasses of a particular polygon, examples of all the various subclasses should be included in the set of polygonal cutouts. For example, several cutout examples having the shape of equilateral, isosceles, and scalene triangles should be included. Also important for inclusion are examples of "typical" polygons, as usually found in textbooks, and "atypical" polygons, such as a "thin" triangle (see fig. 27.1). The greater the variety of polygon cutouts, the greater the potential to learn about students' reasoning. To help the teacher assess the students' reasoning, the students can be asked to record on paper their sorting rationale, as well as to draw and name the different figure categories.

Example of students' thinking. When sorting figures like those shown in figure 27.1, students used various rationales. Many students used the number of sides as their criterion, which indicates reasoning based on attributes. For these students, assigning both the Polygon Definition and Concept Card tasks would be appropriate because these tasks encourage students to begin to formulate definitions of various polygons' parts and properties. These tasks are described in the following sections. Students could also be asked to sort their polygons into various subclasses. For example, the students could be asked to further sort their quadrilaterals to discover some relationships among the figures. Another activity would include the Name and Property Cards task (Fuys, Geddes, and Tischler 1988). This task asks students to match name and property cards (see fig. 27.2) with the groups of polygons sorted previously. Because students may have sorted a particular polygon into subclasses, it is important to include several copies of each of the different name and property cards. For example, the "All

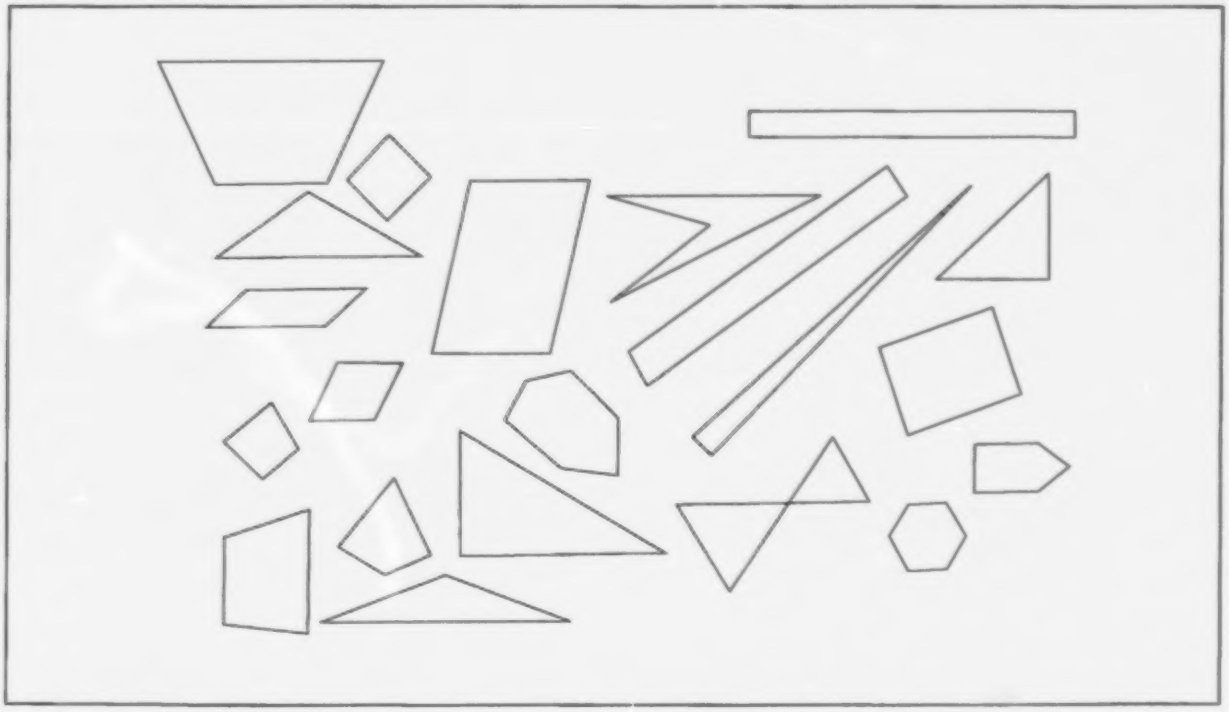


Fig. 27.1. Figures used for the Polygon Sort task

angles are congruent" card would be appropriate for three different groups: squares, rectangles, and equilateral triangles.

The Polygon Definition task

Task description. This task helps identify students' conceptions or limited conceptions about polygons. Students are asked to define a given polygon and provide a rationale for their definition (adapted from Burger and Shaughnessy [1986]). To begin, each student is asked to write a definition for a given geometric figure, then the teacher leads a discussion of the definition. To help students' clarify their thinking, the teacher draws a figure to match the students' definition. If the teacher's drawing does not produce the intended result, then students can modify an inappropriate or incomplete definition. This task is most appropriate for students who have begun to reason using a figure's attributes. It requires students to focus on a figure's parts and properties while reaching group consensus on the minimum amount of information necessary to define a given geometric figure. Students can then complete the Name and Property Cards task described previously (fig. 27.2).

Triangle
Rectangle
Rhombus
Octagon
Equilateral Triangle
Kite
Trapezoid
All of its angles are congruent
Has one pair of parallel sides
Closed
All of its sides are congruent
Concave
Has two pairs of parallel sides
Acute Triangle
Has opposite sides congruent
Has no congruent sides
Has four congruent sides but does not have to

Fig. 27.2. Information in the Property and Name Cards task (Fuys, Geddes, and Tischler 1988)

Example of student's thinking. When fifth-grade students were asked to define a rectangle, one student stated that "it has four sides. It's a polygon and all four sides would be right angles, and has two sets of parallel lines and the only difference is two lines are shorter than two other lines." This definition describes the prototypical rectangle, since it includes properties not necessarily held by all rectangles. From this information, their teacher could follow up by confronting this misconception, perhaps by constructing a Concept Card using a wide variety of rectangles to help students discover the diversity of rectangles.

The Concept Card task

Task description. Students examine a card that displays examples and nonexamples of a geometric figure that has been given a fictitious name, say, *quirp* (adapted from Geddes and Fortunato [1993]). The students are then asked to identify quirps from a set of various figures and to give a definition of a quirp. An example of a concept card, along with one student's work, can be found in figure 27.3. This task is an "effective way to familiarize students with a concept and to make them aware of its distinguishing characteristics" (Geddes and Fortunato 1993, 204). It also requires students to move beyond reliance on rotely acquired facts about standard geometric figures. Teachers can construct different concept cards for various types of polygons and their subclasses, giving each type of polygon a different, fictitious name. A concept card can be made by first choosing one type of polygon and giving it a new, fictitious name; drawing examples of that polygon and any polygons in its subclasses; and then constructing nonexamples to include classes of polygons that distinguish other types from the types shown on the concept card. For example, if constructing a concept card for rectangles, examples should include both rectangles and squares; the nonexamples should include trapezoids, parallelograms, rhombuses, and kites, as well as quadrilaterals that do not fit into any of these classes, such as a chevron. An additional example of a concept card can be found in Geddes and Fortunato (1993, 208). As a follow-up activity, students can also be asked to construct their own concept cards for various types of polygons. Such a task can help a teacher determine to what extent students understand a particular type of polygon, as well as its relationships with other types of polygons.

Example of student's thinking. The student completing the concept card in figure 27.3 was able to successfully identify and draw quirps, or parallelograms. However, he had more difficulty defining the shape fully. Since his difficulty centered on writing the definition, his teacher could follow up this task with the Polygon Definition task described previously. This extension activity could help the student focus more intently on the crucial attributes and properties of parallelograms.

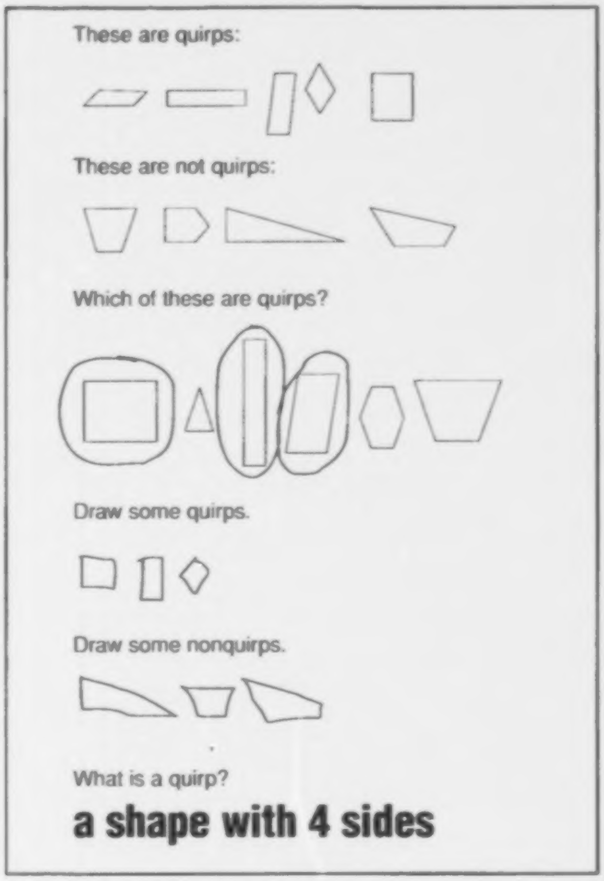
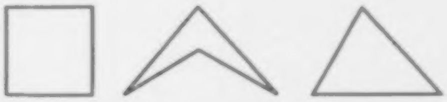


Fig. 27.3. Items examined as part of the Concept Card task

Action Research Ideas

To consider the ideas discussed here, ask students to compare these three polygons:



The information gained from this particular trio can be helpful in determining whether students are using resemblance as the criterion for reasoning about polygons. On the basis of students' thinking elicited from this task, teachers can choose from among the other tasks described here. Several action research activities are suggested:

- After students have completed one or more tasks, teachers can consider and describe how students' backgrounds, characteristics, or interests may relate to their interpretations of the various tasks. Investigating how results from task assessments relate to individual student characteristics could help teachers choose possible remediation or extension activities.
- Teachers can examine how results from the described tasks relate to student performance on other mathematical assessment tasks. For example, descriptions of students' performance on geometric constructions using a geoboard could be compared with results on the Polygon Definition task to assess differences between visual and concrete abilities.
- Teachers may develop a miniature version of a case study for a student whose task assessment indicates that he or she is above or below a "norm." They could document change over time after completing a preestablished course of instruction.

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Educating Hannah: It's a What?

by Sir Wilfred H. Cockcroft and John Marshall

Bill Cockcroft, who resides in Oxfordshire, England, is a retired mathematics professor who maintains an interest in the reform of school mathematics. He was professor of pure mathematics at the University of Hull, Hull, England, and later vice chancellor of the New University of Ulster in Northern Ireland. He served as the chairman of the Nuffield Primary Mathematics Teaching Project in the 1960s and as the chairman of the committee of inquiry that produced the report "Mathematics Counts" (1982), the equivalent in England of the NCTM's Standards in the United States. John Marshall, JohnMarshall@compuserve.com, teaches in the college of education at the University of South Florida, Saint Petersburg, FL 33701. He recently worked with the Minnesota Framework Group as consultant, contributor, and keynote speaker on the topic of data handling. His mathematical interests include games and applying mathematics.

IT'S A what?" asked Hannah in a tone of disbelief. "It's a thick circle," repeated her older sister, Sarah.

Hannah and Sarah were naming shapes in the supermarket while their father did the shopping. They had just seen a container of cheese spread, and Hannah had asked Sarah to tell her its mathematical name.

"There's no such thing as a thick circle!" said Hannah, trying not to laugh. "You must be joking. Who told you that? There's no such shape. The cheese-spread container is called a cylinder, isn't it, Dad? It is not a circle because you can handle it. It will stack. You cannot do that with a circle."

At her school, Hannah is experiencing a reformed-based mathematics curriculum in which she has been taught, from an early age, the mathematical properties and names of different geometric shapes. She and her friends had played initially with cubes, rectangular prisms (cuboids), cylinders, and triangular prisms, in two sizes and four colors, to find out all about them. (See fig. 28.1.) They found out what the shapes felt like, which would roll, which would slide, and which could be used to build similar shapes. Later they would come to see that all shapes that would stack yet remain the same shape, only taller, are known as prisms.

To really get to know these basic solid shapes, Hannah and her friends had been encouraged to collect examples among packages that they found in stores and put them with their classroom sets. They also used these packages to make models of such things as trucks, railway locomotives, and buildings. Making puppet characters was also good fun, with Cuthbert Cuboid being a particular favorite. Nobody actually collected a circle from the supermarket.

Fig. 28.1.

For Hannah, a circle is a plane shape—that is, two dimensional—and cannot have any "thickness." If it had thickness, it would have to be a solid shape. In her work in mathematics, Hannah had discussed the properties of plane and solid shapes at great length with her teacher. She had learned to try to be as precise in her work with shapes as she tried to be in her work with numbers. Sarah clearly had not been so fortunate.

"I'm sure I was taught to call that kind of shape a thick circle," said Sarah. "I can remember sorting colorful plastic shapes into large ones and small ones, into thick ones and thin ones, and circles, definitely circles. I am sure a cylinder was never mentioned."

"Well, if you were taught that," said Hannah, "how thick do your thick circles have to be before they become cylinders? Or put the other way 'round, how thin does a cylinder have to be to become a circle? Have you ever thought about that?"

"Not really," replied Sarah, "but I know there are thick circles. Forget about the cheese-spread container. How can pennies be cylinders? They've got to be thin circles."

"Oh, gosh!" said Hannah, "You've been completely misled if you think there are thick and thin plane shapes. I can see that I will have to give you a lesson about shapes. Wait until we get home."

A Penny for Your Thoughts

As they came through the checkout, Hannah asked their father if he would exchange a dime that she had in her pocket for ten pennies. When they got home, she sat at the kitchen table and made a cylindrical pile of the coins, carefully standing them upright on a piece of paper laid on the table. It was then that their younger brother Andy joined them. Andy was nearly seven and at the same school as Hannah.

"Andy, what mathematical name would you give to the shape of this pile of pennies?" asked Hannah.

"It's a cylinder, of course," replied a confident Andy.

Before Hannah could say anything else, Sarah interrupted. "Well, when does a cylinder become a circle?"

Sarah began to smile to herself as Andy took the coins off the pile, one by one from the top. She felt sure that she was going to be proved right and that Andy would stop at the last coin and declare it to be a thick—or thin—circle.

As he carefully removed each coin, Andy told them that what was left was a cylinder. It was always a cylinder. As he came to the last coin, he said, "Now you have to keep taking the top one off in your mind. When you come to the last one in your mind, put your finger on it and draw 'round it. The 'footprint' will be a circle."

Dad then joined in and said, "And all the coins Andy took off the pile were cylinders, as well. The only way you can get a circle from a cylinder, Sarah, is by looking at an end head on and trying to forget the shape 'behind' it. You can only really do that in your mind."

Andy added, "You know, when Mom slices up a cylindrical loaf of bread, the slice you get is always a cylinder. Even the very, very, very, very thin ones. There's no such thing as a slice with no thickness. You couldn't eat it. We talked about that in school."

Hannah grinned, for she was once in that class.

"All right," sighed Sarah. "But I'm sure I've heard Mom talk about cutting out a circle when she is rolling out her pastry to make a pie bottom. I'm sure that her circle of pastry was at least as thick as a penny, if not thicker."

"Yes, I've heard her too, and we all know what she meant," said Dad again. "To be mathematically correct, she really ought to have talked about cutting out a circular cylinder. In fact, to be even more accurate, she should have talked about cutting out a model of a circular cylinder. In fact ..."

"Why make all this fuss?" interrupted Sarah.

Mathematics in the Mind

We have to make a fuss and be extra careful in teaching mathematics. Children need to learn to use care in their mathematics lessons. If Hannah's teacher had been at the kitchen table with Hannah, Sarah, and Andy, she might have added that mathematics is all about exact things and exact thinking. She would have agreed that what Mom calls the shape of the pie does not alter its taste, any more than not weighing the flour and the butter to the nearest tenth of a gram would. Doing practical things in an approximate way helps us understand what is going on in an exact way in mathematics; in return, we can use these approximations all the time in our ordinary lives. Mathematics itself is about exact and precise ideas, about concepts and abstractions that are in our minds. Although mathematics can be used to solve problems in our real world, and things in the real world can be used to help us think about mathematics, its ideas are not things that we can see and touch. Mathematics is in the mind, in our thoughts.

When her class started to look around and learn about shapes, Hannah's teacher was aware of another kind of problem that arises in translating the mathematics of shapes into work with models of those shapes. In dealing with models of numbers, the teacher can show that a set of, say, two horses is exact in the sense that precisely two horses are in the set, no more and no fewer; not "roughly" two, but exactly two. But in making and drawing geometric objects, the teacher recognizes that models that are made or drawn can be only approximately what is being discussed in the mathematics; yet the models are given the same names as the "exact" mathematical objects: cubes, rectangular prisms, triangular prisms, cylinders and squares, rectangles, triangles, circles. The models are approximate because all measurements are approximate, so equal lengths can never be equal in a model.

Hannah's teacher saw no need to make a great song and dance in class about this discrepancy; she just made sure that when they measured and made their models, Hannah and her friends were aware of the "exactness" of a mathematical idea and the "roughness" of a model of that idea.

Three-Dimensional Models First

Hannah's teacher had first introduced three-dimensional models because these models could be made in a real sense: whatever else these models might be in their roughness, they really were three-dimensional. Indeed, young children who were at the concrete operational stage of their development could handle them. The teacher called the shapes by their mathematical names: cube, rectangular prism, cylinder, and so on, only after the children had handled them, played with them, and generally got to know them. She produced models of the shapes' two-dimensional faces by having the children print outlines of the end faces of their solids and make "footprints" by stamping the shapes on paper. (See fig. 28.2.) The teacher developed this activity because at such a young age, the children's coordination was such that they could not get a good plane-shape image by drawing around the base of each solid shape. However, they could hold a cylinder and stamp a clear model of a circle. This process was important to her because she believed that to understand "cylinder," the children had to understand "circle," and vice versa.

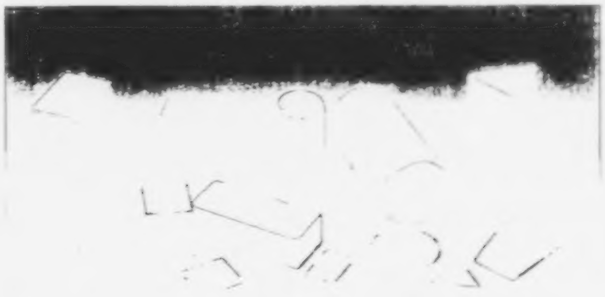


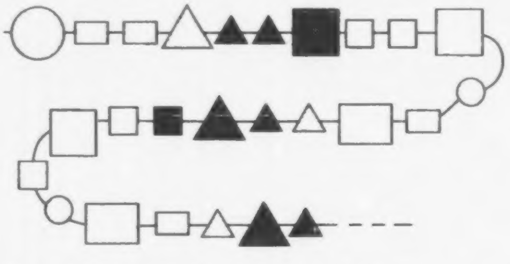
Fig.28.2.

Hannah's teacher used the mathematical names for the shapes of the faces even though she recognized the approximate nature of the pictures that the children produced. Next, she challenged the children to make a truly two-dimensional model. Each time they thought that they had made one, she convinced them that it was really three dimensional, usually by making copies and putting them together to produce a more visibly thick shape. For example, she painted a supposedly two-dimensional rectangle every day until the paint stood out from the surface of the paper. Hannah and her friends had to admit that this "thick rectangle" could not have started off with no thickness and agreed that it had begun as a very, very, very thin rectangular prism of paint! As if to make the point, the paint sometimes would flake off, leaving something for the children to pick up off the floor.

Hannah had created her own explorations of solids and plane shapes. She made sets of various solid shapes by putting her cereal boxes, packets of gum, an eraser, some post cards, and so on, with the rectangular prisms from her geometrical shapes. She also made sequences with both her solid and plane shapes. (See fig. 28.3.) She included size, color, and number in the rules for her pattern as she printed more complex sequences, challenging her friends to print the next three or more shapes. (See fig. 28.4.) Her teacher added another task by asking them to predict, and verify, what, for example, the thirty-fifth shape would be. Beyond counting and shape recognition, Hannah had learned to argue logically about patterns of shapes, always using correct mathematical language to describe what she was doing.

Fig.28.3.

Print and Color the next three shapes.



What would the 35th Shape be in the pattern?

Fig. 28.4. One of Hannah's patterns. Note the shape's sequence of large, small, small, large, small, small

Conclusion

For Hannah, the world of formal measurement with its rules for finding lengths, areas, and volumes of geometric shapes is yet to come, but she has learned and understands the right and proper rules for the language of shapes and their manipulation. (See fig. 28.5.) Throughout the rough modeling and approximation, the precision of the underlying mathematical theory has been exposed. Hannah's teacher has rightly taken to heart this maxim: "The best time to learn mathematics is when it is first taught; the best way to teach mathematics is to teach it well the first time" (National Research Council 1989, 13). Misconceptions are not a part of Hannah's mathematical experience, and for Sarah, thick circles no longer exist.

Students should recognize, describe, and model shapes and relationships and represent and reason about them in increasingly abstract ways.

- Know, recognize, sort/classify, compare, describe, and name (using correct terminology) familiar 3-dimensional shapes in their world.
- Explore ways to combine, separate, and change familiar 3-dimensional shapes (e.g., make models using boxes or other "junk" material of known 3-dimensional shapes).
- Know, recognize, sort/classify, compare, describe, and name (using correct terminology) illustrations of familiar 2-dimensional shapes in their world.
- Explore ways to combine, separate, and change familiar 2-dimensional shapes (e.g. draw pictures involving familiar 2-dimensional shapes in their world).
- Explore the concept of a prism.
- Understand the difference between 3-dimensional shapes and a 2-dimensional shape (e.g., explain the difference between a "cylinder" and a "circle").
- Play logic games involving properties of everyday solid and plane shapes (e.g. classify by 1, 2, or 3 attributes using Venn diagrams; complete "one difference" chains) using correct mathematical language to describe decisions.
- Create sequences using shapes and various attributes, and explain their sequences in words using the correct mathematical language.

From SciMath^{MN} *Minnesota K–12 Mathematics Framework;* used with permission.

Fig. 28.5. Shape, Space, and Measurement Components: Primary K–2

Hannah and Sarah are going to the supermarket with their father and have decided to play "name that shape" again.

"I am going to take a pencil this time so that I can write down the names of the shapes," announces Sarah.

"I wonder if she knows the name of the shape of her pencil!" thinks Hannah.

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National Research Council (NRC). *Everybody Counts.* Washington, D.C.: NRC, 1989.

SciMath^{MN}. *Minnesota K–12 Mathematics Framework.* Saint Paul, Minn.: SciMath^{MN}, 1998.

The solid shapes and printing materials used in the experiences described in this article are supplied by World Class Learning Materials, 111 Kane Street, Baltimore, MD 21224, (800) 638-6470. Further details about SciMath^{MN} Minnesota K–12 Mathematics Framework can be obtained from Debra Rose at (612) 296-4058.

The Role of Definition

by Jane M. Keiser

MUCH of formal geometry is based on unambiguous definitions that can be used to develop and prove theorems. Is it possible that such clear and precise definitions can hinder conceptual understanding? Just what role should definitions play in the mathematics classroom? Should students be told definitions? Or should they be asked to develop them? In this era of educational reform, we are emphasizing student involvement in discourse. The NCTM's *Professional Standards for Teaching Mathematics* states,

> When students make public conjectures and reason with others about mathematics, ideas and knowledge are developed collaboratively, revealing mathematics as constructed by human beings within an intellectual community. (NCTM 1991, 34)

When students are heavily involved in classroom discussions and negotiating their own meanings, at what point should specific mathematical definitions enter their conversations? The *Professional Standards* lists "deciding when and how to attach mathematical notation and language to students' ideas" (p. 35) as one of the tasks that teachers face as they attempt to promote student discourse. This decision often occurs on the spur of the moment, depending largely on the complexity of the concept being investigated. Textbooks that once contained glossaries are now becoming less structured in their approaches to definition, leaving the creation of definitions up to the teachers and students.

This article addresses the role that mathematical definitions can play in the middle-grades mathematics classroom, and it specifically focuses on the concept of *angle* as it was introduced to sixth-grade students. This issue arose during a dissertation study. The curriculum used to teach students who were involved in the study was the Connected Mathematics Project (CMP) (Lappan et al. 1996), which was funded by the National Science Foundation. During this research, I had the opportunity to observe instruction in which the process of introducing geometric definitions took on a very different flavor from that in traditional instructional approaches. Because the students were involved for several weeks in negotiating the meaning of *angle*, they had numerous opportunities to clarify their own understanding and confront basic issues raised by other students as they worked through geometric activities.

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Traditional Approach to Teaching the Concept of Angle

Consider for a moment a traditional approach to teaching the concept of angle. In two older sixth-grade textbooks, an angle is defined simply as "two rays with a common endpoint" (Orfan et al. 1988; Hoffer et al. 1991). When I taught seventh grade in the early 1980s, I spent a few days presenting students with a similar definition of angle and teaching other ideas related to this concept. We focused on learning related vocabulary, such as *perpendicular, complementary, supplementary, acute, right, obtuse,* and *straight angle.* When I presented pictures of angles to students in their textbooks, I asked them to name the angles correctly, determine the kind of angle shown, find the complement of an angle, and so on. The last few activities involved measuring angles with protractors.

In my five years as a middle school teacher, I never imagined that the concept of angle was difficult for students. I had what I thought was a pretty good "concept image" (Tall and Vinner 1981) of an angle and could not imagine that the students might be struggling to form their own images of an angle. Inherent in this traditional approach, however, is the expectation that students will comprehend mathematical meanings based on abstractions from the real world. For my students, this abstraction might have been easier if they were first presented with concrete representations of rays, points, and planes, which were lacking in my instruction. I also emphasized geometric vocabulary in my instruction, although I made little effort to require that students use this language to communicate thoughts or ideas. Instead, I provided definitions early and asked students to use these definitions for the sole purpose of identification.

This practice of clarifying terms at the beginning of instruction was central in Harold Fawcett's dissertation study, "The Nature of Proof," which became NCTM's thirteenth yearbook. He wrote,

> As the first step in leading the pupils to understand something of "the nature of proof," it seemed important that they should recognize the necessity for clarity of definition in all matters where precise thinking is essential. To understand how the vagueness and ambiguity of ordinary words lead to serious errors in reflective thought is to appreciate the importance of clearly defined concepts in any technical vocabulary. (Fawcett 1995, p. 30)

The instruction in Fawcett's study, however, involved ninth, tenth, and eleventh graders who were beginning their first formal geometry classes in high school. The teacher in Fawcett's study involved his students in "nailing down" definitions early in the course so that these definitions could serve as a foundation on which to build an axiomatic system. In my own teaching, I assumed that my middle school students were just as ready for such abstract definitions. My subsequent observations of a CMP classroom changed my thinking completely.

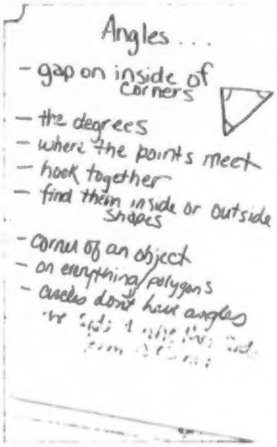
An Alternative Approach to Teaching the Concept of Angle

The CMP writers chose an alternative approach in their first presentation of the concept of angle to sixth-grade students. In their "Shapes and Designs" unit, angle is not defined. Instead, the writers chose activities that would build on the students' understandings before providing a formal definition. They present angle as a concept that can be thought of in three different ways: as a wedge, a turn, or an intersection of two lines. Students explore these three representations by using manipulatives that highlight the various features. For example, their hingelike angle rulers open and close, allowing them to observe the rotational features of an angle. Shape sets and pattern blocks give students the opportunity to view angles as wedges. Polystrips—plastic strips attached to each other with removable metal brads—show the view of an angle as the intersection of lines. The investigations, like the manipulatives, are varied so that students can encounter many features of angles.

I observed the teaching of "Shapes and Designs" in two Michigan sixth-grade classrooms in which the CMP materials had been used in a pilot program for two years. The teachers, therefore, were familiar with the content and the underlying CMP philosophy of student-centered learning. Both female teachers, referred to as T1 and T2 in the following transcripts, were adept at facilitating discussion and were experts at getting students involved.

Students' concept images of angle

The teachers were committed to involving the students as much as possible in discussions. They were also firm in their beliefs that students should participate in defining mathematical ideas. Although the teachers occasionally defined some of the mathematical terms, more often they involved the entire class in generating and recording descriptions of more abstract concepts. Both teachers used this approach in teaching the concept of angle. For example, the photograph in figure 29.1 resulted from one class's initial responses to the question "What is an angle?"



Angles . . .

- gap on inside of corners
- the degrees
- where the points meet
- hook together
- find them inside or outside shapes
- corner of an object
- on everything/polygons
- circles don't have angles

Fig. 29.1. Chart listing students' initial descriptions of angle

Throughout the five weeks of geometry instruction, the students were never given a formal definition of angle. They were, however, given several opportunities to describe an angle and its qualities. Interestingly, students' initial descriptions usually centered on one aspect of the angle—the vertex, the rays, or the interior.

Emphasis on the vertex. When asked, "What is an angle?" Adele responded, "another word for corner," and Dixie thought that an angle was the "corner of an object." Mindy said that angles were "where the points meet." After Kay had first pointed to the edges of the angle, she changed her mind and, pointing to the vertex, said, "right here, where it meets."

Some of the students' writings in their journals also suggest that they equated the vertex of the angle with the angle itself. For example, Joe's and Mandi's entries in figures 29.2 and 29.3 use the words corner and point. In fact, even when the class was finalizing its descriptive list near the end of the unit, Dave still struggled to distinguish between points and angles. He contributed the following to a discussion in which the teacher was revising the students' ongoing list of descriptions of angle.

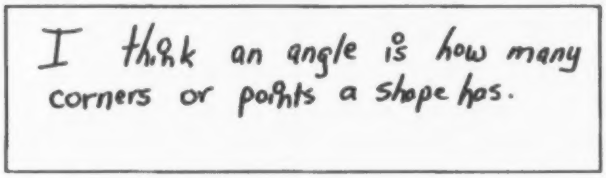

I think an angle is how many corners or points a shape has.

Fig. 29.2. Joe's journal entry

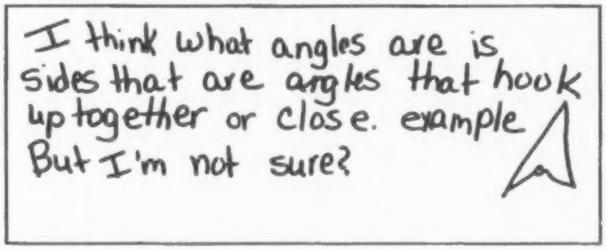

I think what angles are is sides that are angles that hook up together or close. example
But I'm not sure?

Fig. 29.3. Mandi's journal entry

Dave: I think you should either say that or say they form a point which is an angle.

T2: An angle and a point are the same thing?

Group: No!

Dave: [Dave then agrees with the others.] Not really.

T2: This is a point [pointing to the vertex]; is it also an angle?

Emphasis on rays. Some students focused on the rays when they described angles. In fact, many students used the words side and angle interchangeably, just as the students in the previous section confused point with angle. (See Tamara's journal entry in fig. 29.4.) As a consequence, some of these students misidentified the length of the rays as the measure of an angle. For example, Ricci wrote the journal entry in figure 29.5 and made similar comments in class.

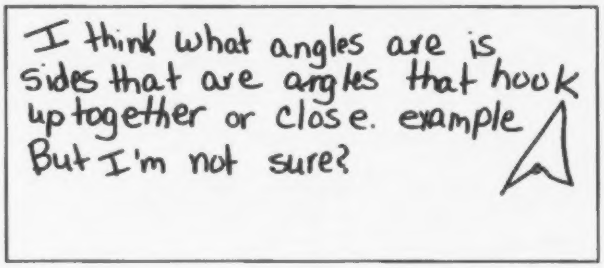

I think what angles are is sides that are angles that hook up together or close. example
But I'm not sure?

Fig. 29.4. Tamara's journal entry

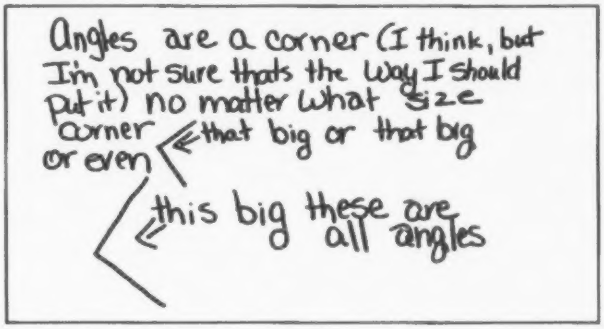

Angles are a corner (I think, but I'm not sure thats the way I should put it) no matter what size
corner that big or that big
or even
this big these are all angles

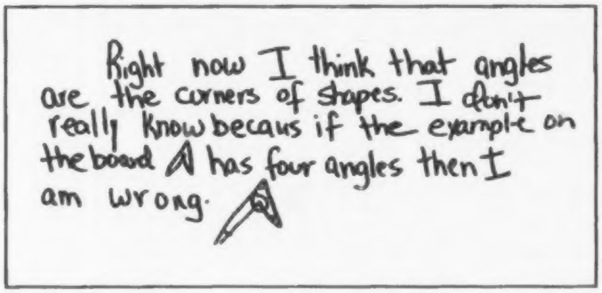
Fig. 29.5. Ricci's journal entry

Emphasis on interior. Abe said that an angle is "a measurement measuring the direction from a point to another point." Meg stated that angles were "the width between the two lines." Although this measure seems to indicate a linear measure across the interior of an angle, other students spoke more about the area between the rays. For example, Raymond defined angle as "the amount of space between two rays." Dave said that an angle was "the gap on the inside of a corner" and "the inside of the corners." Tess argued, "It's around the point. Because some of those corners are on the inside. But if you follow around the point, some [angles] can be inside and some can be outside. So maybe angles are just like the gap around the point." These students seemed to have more of a rotational idea in mind and, when using such words as gap and space, they seemed to be describing the area between the two rays.

Student arguments

The students not only gave a wide variety of descriptions of angle but also argued about inclusion and exclusion of certain kinds of angles. In developing a deeper understanding of such a concept as angle, it is important to decide what an angle is and what an angle is not. Class discussions solidified the boundaries for the concept images that were being formed. For example, Ricci posed a question to the whole class about the inclusion of curves. She wondered whether the letter S contained angles and thought that it might contain two. Although similarities are found between curvature and the measure of an angle, Ricci struggled to distinguish between the two.

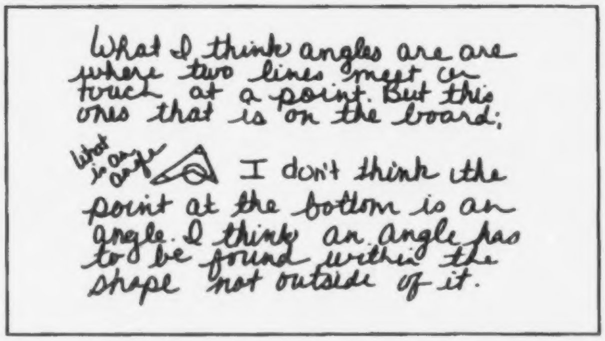
On another day, when Kay made a concave quadrilateral out of polystrips, the entire class began to debate whether the concave interior angle was considered an angle. Brenna reflected on the question in her journal (fig. 29.6).



Right now I think that angles are the corners of shapes. I don't really know becaus if the example on the board has four angles then I am wrong.

Fig. 29.6. Brenna's journal entry

Students continued to share their thinking about angles found in the interiors of polygons. If an angle was not on the interior of a polygon and did not have a measure smaller than 180 degrees, then students struggled to identify it as an angle. At first, Kelsey agreed with Brenna that the angle formed in the concave quadrilateral was not an angle and gave the response shown in figure 29.7 to support her argument. These arguments were never resolved by the teachers. Instead, throughout the month the teachers raised these issues for the students to ponder. By the end of the five weeks, more students were making comments, similar to Tess's ("some [angles] can be inside, and some can be outside"), that allowed for angles to be found inside or outside of polygons.



What I think angles are are where two lines meet or touch at a point. But this ones that is on the board;

What is [illegible] I don't think the point at the bottom is an angle. I think an angle has to be found within the shape not outside of it.

Fig. 29.7. Kelsey's response to the concave equilateral discussion

Angles with measures of 180 degrees and 360 degrees were also difficult for at least one student to accept as angles. Claire had always thought of angles as being composed of lines and corners; because the corner, or vertex, is not observable with angles of these measures, she excluded them from her concept image of angle. Claire commented,

> I saw an angle as having a vertex point and then two lines going different ways. And I know they are going different ways, but you can't tell where the vertex is and how they are going different ways or if they're sort of doing the same thing. . . . All I think what the angle really is, is two lines put together on their corner, propped up on their corner, but this has no corner to be on. I don't understand how an angle can just be a turning motion and not two lines.

In fact, seeing the "turning motion" inherent in the angle concept was mostly missing from students' original descriptions. It is certainly likely that the curriculum unit's final section, devoted to Logo programming, would have reinforced a turning approach to the concept. Logo gives students the opportunity to program a turtle's path and its turns as it travels across the screen. However, the teachers believed that after five weeks, they needed to move on to other important topics in the CMP curriculum. Even without this instruction, the turning notion was addressed as the students argued with one another, and interestingly, in Claire's final report, part of her description of an angle contained the phrase "two lines that meet where a turn is measured by degrees."

Another student discussion centered on an investigation of Amelia Earhart's last flight. The students were given a map of three locations: Lae, New Guinea (where Earhart was last seen); Howland Island (her destination); and Nikumaroro (the island where it is believed she crashed). They were then asked to use their angle rulers to see how far off course Earhart had been, which was about 7.5 degrees. The goal was for students to discover how a small error in measurement can bring about disastrous consequences, depending on the situation. Claire, however, raised a different issue when she contributed to the following discussion:

T1: About 7 1/2 degrees, okay. Was she off very much as far as the angle is concerned?

Group: No.

Claire: But that's like on a small scale though. In real life. it would be really big.

John: It would be like 200 degrees.

T1: What do you mean by that?

Claire: That's a small map. If you made the map bigger, if you blew it up bigger, it would be a different distance, because [the islands] wouldn't be so scrunched together. It would be farther apart. And if it was real life, it would be a really big angle.

Claire's comment raised a question that was discussed periodically over the course of the unit: Do angles get bigger the farther you move from the vertex? The answer to this question was definitely not obvious to the students. As the question was discussed, many of the difficulties that students had with their original descriptions of angle came to the surface and were addressed. One student was finally able to convince Claire and the others by pointing to the hands of the clock on the wall and the hands of the watches on their wrists. At the time, the angle made in both places was a right angle. The student argued that even though the hands on the clock and watches were different lengths and the area between the two was very different, the same 90-degree angle still appeared on both.

This kind of communication among students helped each student to refine his or her own meaning of angle. Students who had focused too heavily on the length of the rays learned that an angle that could be drawn in their books had the same measure as the real-life angle that spanned an ocean. Students who had thought of an angle as being a linear distance drawn somewhere between two rays learned that linear distance increases as one moves farther from the vertex, but rotational distance stays constant. Consequently, students involved in these conversations came away with some ideas about how angles are related to similarity and slope.

Discussion

These vignettes confirm that angle is a very complex concept for students in the sixth grade. A simple declaration of a mathematical definition—that is, two rays with a common endpoint—might prohibit students from further exploration of this abstract concept. The definition alone cannot address the questions that were raised by the students in these classrooms. In fact, all definitions, by their precise, exclusive nature, may limit exploration and prohibit questioning because they offer an authoritative "last word." After hearing and considering their classmates' differing views, however, the students that I observed were more prepared to construct a broader, more complete image of angle that could be flexible enough to fit the many mathematical situations in which we use angles.

During a listserve discussion for mathematics educators, Alan Schoenfeld commented,

> There's a classic mathematical saying, "We make the definitions so that we can prove theorems." It's my impression, informally, that that does happen a lot—and that in the reformulation of definitions so that they can do work for us, we tend to lose some of the intuitions that gave rise to the definitions in the first place. (7 February 1997)

By formally defining concepts to middle school students who are not yet prepared to use definitions for the purpose of formal reasoning, might we be limiting their understanding of these concepts? The initial struggles of the students I observed to communicate meaning and their original insistence on only one concept of angle suggest that mathematics instruction in the earlier grades should attempt to build broader concept images before formal definitions are introduced.

As stated previously, teachers have the task of "deciding when and how to attach mathematical notation and language to students' ideas" (NCTM 1991, p. 35). In contrast, the *Professional Standards* suggests that the students' role is to "initiate problems and questions; make conjectures and present solutions" (p. 45). It seems only natural that the classroom community would more honestly reflect the way that definitions are developed if the students were actively involved in the defining process instead of being recipients of existing definitions. The sixth-grade students that I observed formed an intellectual community in which they challenged one another's thoughts and images to refine their own ideas; because their conflicts were raised and dealt with, each student developed a much broader and more flexible understanding of angle.

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Measurement

Instructional programs from prekindergarten through grade 12 should enable all students to —

- *understand measurable attributes of objects and the units, systems, and processes of measurement;*
- *apply appropriate techniques, tools, and formulas to determine measurements.*

The two articles illustrate the importance of early, informal experiences with the concept of measurement. Ruggles and Slenger describe a kindergarten–first grade project involving an introduction to the attributes of height and weight. Students, with parental and teacher help, constructed dolls on the basis of their individual birth lengths and weights. Year-long activities with these dolls helped teach the meaning of measurement and provided children many opportunities to make and reflect on comparative measurements.

The article by Kubota-Zarivnij presents the results of a second-grade Father's Day project that combined investigations with both informal and formal measurement units. Students completed booklets describing how they measured the attributes of length, perimeter, and area of one of their fathers' shoes. The project began with the reading of a story about shoes and also included classification, measurement, and recording activities on the assembled classroom collection of all the fathers' shoes.

The "Measure Me" Doll

by JoLean Ruggles and Barbara Sweeney Slenger

> Students retain best the mathematics that they learn by processes of internal construction and experience.
>
> —*Everybody Counts: A Report to the Nation on the Future of Mathematics Education*

AT THE McKinley Thatcher School, the teacher of a kindergarten–first-grade class usually starts the school year with a "Me" unit to get to know the students. Their ages range from five to seven years, and the teacher keeps the students for both years. The "Me" unit, then, helps students to get acquainted and enables the teacher to note the growth changes from the previous year.

This year, for the "Me" unit, the teacher decided to make "Measure Me" dolls that would focus on mathematics. This simple project of making dolls to represent students' birth statistics turned into months of integrated learning in this classroom. This activity not only built on the students' previous knowledge and experience but also helped the students and the teacher become better acquainted in the process. The most gratifying outcome of the project, however, was the observable development of the students' emergent understanding of mathematics concepts.

Using these dolls could foster any number of mathematics strands, but the measurement strand seemed most appropriate for a beginning-of-the-year project. The *Curriculum and Evaluation Standard for School Mathematics* (NCTM 1989) states that the curriculum should address teaching for understanding of the attributes of height and weight. Estimates of measurement are important, and the tasks using measurement should be ones that the students encounter every day. Making a "Measure Me" doll provided that opportunity.

Before the activity began, the teacher asked the students some questions about themselves to establish their prior knowledge of measurement. When seven-year-old Travis was asked how big he was, he replied, "I'm four and a half feet." He knew that, he told us, because "when I am sick and my mother takes me to the doctor, he measures me and my mom writes it down." When six-year-old Malcolm was asked the same question, he said, "I don't know, but I know I'm six years old, and besides, my brother is smaller than I am because I was born first." Both of these students have some understanding of measurement as it applies to them personally. Travis knows that the "four and a half" is a number that is used to record his height and that the number is getting bigger each time he goes to the doctor. Malcolm knows that for him, being older than his brother makes him bigger; however, he could not numerically label the notion of *bigger*. Developmentally, students who are six to seven years old would have little understanding of units of measurement, but they do know that their height and weight measurements are taken and recorded and that comparisons are made within the family.

One basic set of records was required to make the "Measure Me" dolls: birth statistics. This information is generally available for each student, since a birth certificate is required for entrance into public schools. In the rare event that a student does not have this information, the teacher can suggest one of the following methods to obtain useful figures. The teacher, or the parent, can use the class average (mean) or can use the

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This article was developed from training activities facilitated by Ruggles for the Public Education and Business Coalition, 1410 Grant Street, Suite A-101, Denver, CO 80203.

most frequent (mode) class statistics to supply a set of weight and height figures for that student. Acquiring these statistics was an important transition "home to school" activity and one common to all students.

When the students were asked to get this information, the parents had an opportunity to discuss their child's birth and the importance of that event. Even though the numbers on a birth certificate are very precise, the kindergarten student's understanding would be in gross measurements, such as "more than" or "less than." To help them express this understanding, the teacher used nonstandard units, like connecting cubes, plastic chips, and strands of yarn, to compare lengths. These same estimates were given by the students later when they observed or felt the weight of the doll.

The following description of the steps and lessons in making the "Measure Me" doll shows the students' experiences with numeracy. Just as a child can be an emergent reader, so also can he or she be an emergent numerate learner. Through this sequence of activities, these learners could relate what they already knew about themselves to new situations in which they compared their weights and heights with those of other children.

The Lessons

Materials needed

Baby picture or earliest age available; birth weight and length; four shades of fabric—off-white, cream, darker cream, and brown; five colors of yarn—black, brown, yellow, red, and white; puff paint; scales; permanent markers; birdseed, rice, and steel pellets; large plastic container; funnel.

Enlarging the doll pattern

You will need to enlarge the basic doll pattern shown in figure 30.1 until you have a pattern approximately 19 inches long and 14 1/2 inches across—the dimensions need not be exact. To make a pattern this large, separate it into an upper half and lower half and enlarge each onto 11-inch-by-17-inch paper. Enlarging the pattern to this extent requires multiple passes on a photocopy machine—we found that enlarging it three times at 180 percent each works well. The final step is to tape the two pieces of the final pattern together.

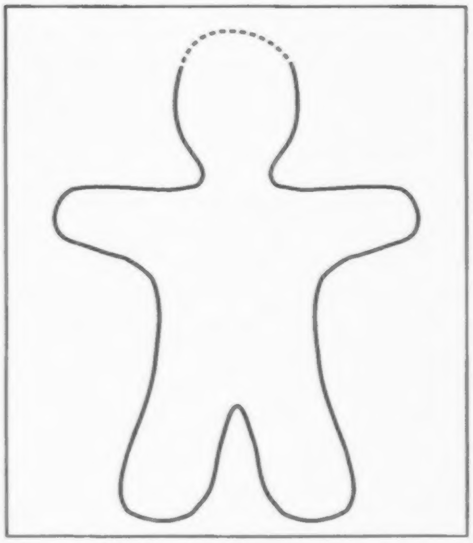

Fig. 30.1. The basic doll pattern.

Helpers

Enlist the help of parent volunteers who have sewing machines. Other parent volunteers can help fill the dolls with the seeds, or rice and pellets.

Procedure

The first task for the students was a homework assignment. They were asked to bring to school a baby picture and their birth weight and length. Birth statistics are all numeracy pieces, and the birthdate is frequently used by most kindergarten and first-grade teachers, so this mathematics activity was a natural extension.

Next, the students chose one shade of fabric for their doll. A parent volunteer then cut and sewed the dolls, making each doll about 19 inches long and about 10 inches wide. The top of the doll's head was left open so that it could be filled.

The students then decorated the faces of their dolls with permanent markers and fabric paint (fig. 30.2). After the faces were finished, each student, with a volunteer's help, filled his or her doll to the child's correct birth weight, using a combination of the filling material. The rice and the seed weigh about the same, and pellets can be added to give the volume more weight. With a funnel, the students poured the seed or rice mixture into the top of the doll's head. Because each student had to weigh the seed and then

Fig. 30.2. A student decorates a doll's face.

Fig. 30.3. A student displays some of the writing inspired by the doll.

fill his or her doll, this operation was ongoing until each student had completed the task. The parents were very helpful. Even though actual birth weight was measured on the scales, the procedure of measuring the rice or seed mixture was done mostly by parent volunteers.

When all the dolls were filled, the students chose colored yarn for the hair. The original plan was to have the students sew the tops, but the teacher realized that this task would be too difficult and not appropriate for this activity. Instead, a parent volunteer sewed the heads of the dolls closed with the "hair" (yarn) attached in the seam.

The Activities

After the dolls were completed, the students played with them. A sense of ownership and identification was quickly formed. Students spent considerable time carrying them, talking to them, drawing pictures of them, reading to them, and moving them about the room. Research tells us that this free time is absolutely necessary for using any manipulative. Only after becoming more familiar with the manipulatives is it possible for students to let the primary focus of their activities be the mathematics concepts to be explored using the manipulative.

The students read stories to the dolls. This activity reinforced the importance of reading and also connected the doll to the student in yet another way. They also wrote letters to and about the dolls. This journal-writing experience was a rich source of expressive writing (fig. 30.3).

The dolls were weighed and reweighed. The students were fascinated by this repetitive activity. The lengths of the dolls were measured with a variety of manipulatives, such as connecting cubes, plastic teddy bears, and dominoes. Because the length was nearly the same for each doll, we did not discuss the differences in the students' results.

When the students held the dolls, they made comparisons with each other and expressed such ideas as "Mine is heavier than yours" or "Your doll doesn't feel as heavy as mine."

Standing and holding their own dolls with the weight tags attached, the students arranged themselves in order from the smallest to the largest birth weight. First they grouped themselves by using the pound measurement, that is, all seven pounders stood together. Then, a second measurement, ounces, was ordered within each pound group (fig. 30.4) and beans

Fig. 30.4. The students display the birth weight of dolls.

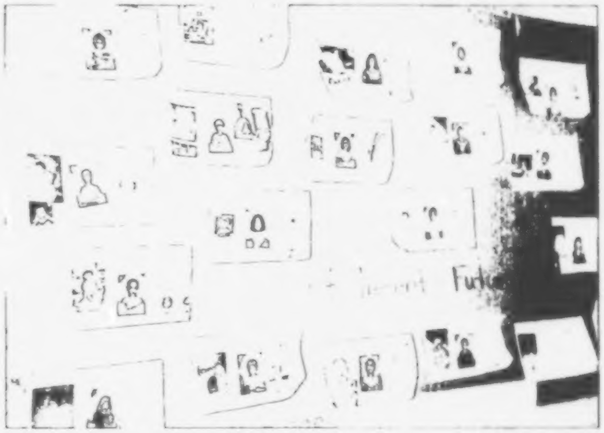

Fig. 30.5. Past, present, and future charts are displayed in the school.

and rice were weighed on a postage scale. The students discovered that the higher number on the scale meant more beans or rice.

After the students completed their arrangement of the smallest to the largest, a parent volunteer placed a long cash-register tape on the floor in front of the students. We wrote the students' initials in red ink to show their placement in the row. Next, we weighed the students on a bathroom scale. We recorded the weights on the chalkboard and arranged the numbers, along with the students' initials, from smallest to largest. Then the students arranged themselves in order according to their present weight, and we noted their placement along the tape with blue ink. The students could then compare their birth weight with their present weight. We heard such remarks as "I used to be fourth and now I'm tenth" and "I didn't stay the same—I moved!"

Finally, we made a chart with these headings: Past, Present, Future. We glued baby pictures under the "Past" heading and recent pictures under the "Present" heading. For the "Future" heading the students drew pictures of what they might look like and what they might be doing next year. We then displayed the chart in the hall for all the students in the school to see (fig. 30.5).

We kept the completed "Measure Me" dolls in school until the winter break, and then the students took them home. Occasionally, the dolls came back to school to visit, and it was interesting to see how they had changed. Some dolls had had operations—rough handling had necessitated stitches in various places. Many dolls came dressed in baby clothes. Some had added physical features, such as ears, navels, or fingernails.

During the months that this project continued, the comments of the students showed their developing number sense. They began to express their ideas about such things as cereal boxes or bags of sugar that they thought weighed two pounds or six pounds. They also made verbal comparisons with other objects: "This box of blocks weighs about as much as Matthew did when he was born."

The one-pound, nine-ounce birth weight of one student born prematurely had caused surprised comments from her classmates. When that student's doll was filled with seed, all the students and even some of the parents were astounded by how different her doll looked and felt. Months later, on this child's birthday, the class discussed her age and what it meant to be premature. Several students immediately remembered and commented on her doll and how much lighter it was than the rest.

The students' understanding of a specific weight, such as eight pounds, ten ounces, was not internalized, but by feel they could tell if an item was closer to eight or to nine pounds. Throughout the rest of the year the students continued to use such comparative vocabulary as "taller than," "shorter than," "almost the same as," "heavier than," and "lighter than" when talking about books and other objects in the classroom. Their notions of size had obviously become stronger.

The uses of the dolls could be many, but two in particular come to mind. First, the collection of data was the initial part of the lesson; therefore, data analysis would be a good next mathematics topic. Second, the possibilities for literacy connections are many, with books like *Much Bigger Than Martin* by Steven Kellogg (1976) and *The Biggest Nose* by Kathy Caple (1985). In fact, many concepts evolved from the basic ideas in three content areas: reading, writing, and mathematics.

This emergent-numeracy project was designed to teach students the meaning of measurement rather than how to measure. All measurement is comparative, and the teacher's objective was for the students to begin to think about how their height and weight compared with those of other children. These lessons gave the students a foundation from which future measurement studies could be launched.

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How Do You Measure a Dad?

by Kathy Kubota-Zarivnij

"Boys and girls, you need to ask your mom for permission to borrow one of your dad's shoes for two weeks. It should be a shoe that he wouldn't use during this time because we don't want him to know that it is missing. You're going to use that shoe to prepare a special Father's Day gift," announced Mrs. Porter to her class of second-grade students. "What if I don't have a dad?" asked a few students. "Would you like to prepare a special gift for an uncle? Your grandfather? Your older brother? A special family friend?" asked Mrs. Vivona.

The children in these classrooms discussed the Father's Day activity and gift with excitement. This gift would include a series of pictorial and written descriptions about their dads' shoes, compiled in a booklet called "How Do You Measure a Dad?" On Friday, the teachers sent home an explanatory letter, outlining the request for a shoe and describing its use in the classroom. Throughout the next week, more than fifty shoes were collected, providing a concrete and relevant data set for students to analyze, classify, and describe mathematically.

Planning the Shoe Investigation

The shoe investigation was designed and implemented by a university mathematics course director and two second-grade teachers. The investigation uses the book *Shoes, Shoes, Shoes* (Morris 1995) and a series of mathematics activities (shown in fig. 31.1) based on the NCTM's Standards (1989) in data management, measurement, number sense and numeracy, and patterning. For example, students related "physical materials, pictures, and diagrams to mathematical ideas" (NCTM 1989, p. 26) using the book, shoe, measurement tools, and record sheets. When students interpreted their shoe graphs in mathematics journals, they communicated their interpretations and descriptions. Further, we assessed students' learning using observation, individual interviewing, and work-sample analysis, including mathematics journals and record sheets.

For the shoe investigation, the students—

- analyzed and described orally and on chart paper the characteristics of shoes depicted in the picture book *Shoes, Shoes, Shoes* (Morris 1995);
- classified the shoes in the book on a hula-hoop Venn diagram;
- classified the parents' shoes on a hula-hoop Venn diagram;
- analyzed the parents' shoes using nonstandard and standard measurement tools, such as cutout paper ants, string, measuring tape, rulers, centimeter-square grid paper, 8 cm × 8 cm cubes, and a triple-beam balance;
- recorded the measurements of the shoes on narrative sheets;
- collected and organized the shoe-measurement data (i.e., length, width, perimeter, area, volume, mass) into class graphs, including vertical and horizontal bar graphs, number-line plots, pie graphs, pictographs, and tree diagrams; and
- interpreted the class shoe graphs in a narrative journal.

Fig. 31.1. Activities for the shoe investigation

In addition, we arranged to have seventh- and eighth-grade students facilitate second-graders' small-

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The author wishes to extend special thanks to Connie Porter and Josie Vivona, second-grade teachers at Saint Jane Frances Catholic School in Toronto, Ontario.

group discussions. Before this activity began, these older students were given an overview of the shoe investigation and assigned responsibilities as facilitators, including recording students' ideas on chart paper, asking questions to promote discussion, gathering needed concrete materials, and interviewing students about their understanding of specific mathematical concepts. We discussed ways that the older students could help the second graders provide more detailed responses through probing questions and statements, such as, "Why did you say that?" "Explain your idea." "How do you know that your answer makes sense?" To encourage the younger students to listen actively to one another, we listed such questions as, "What else can you add to Vanessa's answer?" "What did Carlos say that makes sense to you?"

Qualitative Analysis: "What Makes Shoes the Same and Different?"

To introduce the shoe investigation, the teachers read aloud *Shoes, Shoes, Shoes,* which begins with the sentence, "Shoes, shoes, all kinds of shoes, wherever you find them shoes come in twos" (p. 7) The subsequent pages have photographs and rhyming descriptions of shoes in many real-life situations and cultural contexts. The book includes twenty-eight photographs from around the world and an index that describes each one. For example, in one photograph, Masai dancers leap in thick-soled sandals; in another, firefighters work in black rubber galoshes.

After the teachers finished reading the book to the class, they asked students, "What did you notice about the shoes in this story?" Some of the students said, "The shoes are from different cultures." Another student pointed out, "The shoes protect feet from water in that picture and from snow in that picture." Several students explained, "They're made of different stuff, you know… different material stuff, like leather."

To continue their analysis of the shoes for specific details, students worked in small groups, examining and brainstorming characteristics of different shoes from the picture book. The student facilitators listed the students' thoughts on chart paper and to help generate more ideas, posed probing questions, such as, "What else do you see?" "Why?" "What other ideas are related to this one?"

The students then shared their lists in the large group (see fig. 31.2). After recording some of their ideas on chart paper, the teachers asked the students, "What is special about these shoes?" After synthesizing their ideas, the students identified the following categories using their own words:

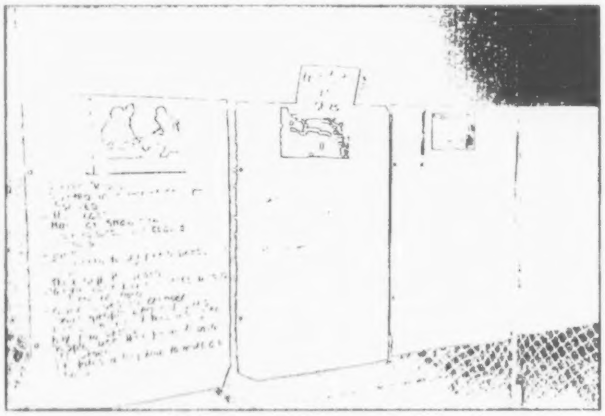


Fig. 31.2. Charts showing characteristics of shoes

- What it is used for (e.g., used for walking in the summer, helps you walk in the snow)
- How it looks (e.g., brown, open toes makes you look cool)
- Parts of a shoe (e.g., has straps, no zipper)
- What it is made of (e.g., leather, straw, rubber)
- Measurements (e.g., old, very heavy, tall)

To further build on this experience, the students worked in groups to sort the pictures of shoes from the book according to the categories they had listed, then shared with, and explained their ideas to, the whole class (see fig. 31.3).

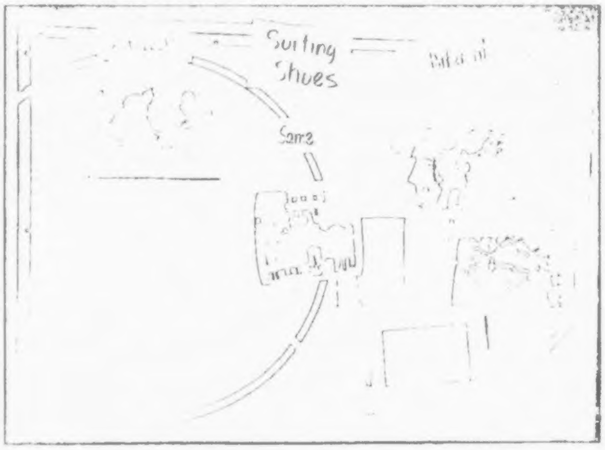


Fig. 31.3. Venn diagram showing shoes sorted into categories

Finally, the students examined and sorted their dads' shoes on the floor, using a Venn diagram made from hula hoops (see fig. 31.4). Students first sorted the shoes according to one attribute from their categories, then two attributes. The student facilitators recorded the sorting characteristics on strips of paper so that the groups had records of their sorting ideas. After about fifteen minutes, the students chose one sorting rule and represented it on the hula-hoop Venn diagram. As the students walked around the classroom, examining and verifying one another's sorting rules, they offered comments, such as, "I agree with that rule." "I don't think that rule is correct; I'm going to check." "They're lucky that they had the shoes to do that rule; all the shoes in our group were almost the same."

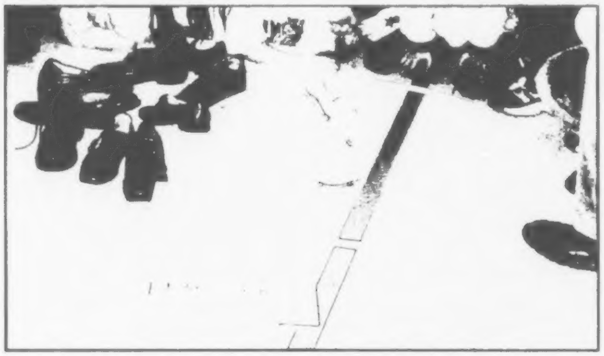


Fig. 31.4. Venn diagram of two shoe attributes

Quantitative Analysis: "How Do You Measure a Dad's Shoe?"

The students continued analyzing their dads' shoes using a variety of nonstandard and standard measuring tools, such as uniform cutouts of ants (see fig. 31.5), string of varying lengths, measuring tapes, rulers, centimeter-square grid paper, and a triple-beam balance scale. As the students explored the relationship between the physical features of their dads' shoes and the measurements, they were "develop[ing] the process of measurement concepts related to units of measurement; making and using estimates of measurement; making and using measurements in problems and everyday situations" (NCTM 1989, p. 51). Most students used the ruler to measure the widest and longest parts of the shoe. The text of the record sheet prompted students to use the paper ants to estimate the perimeter of the shoe. I asked some students, "If you used pencils instead of paper ants, would the perimeter of the shoe be bigger or smaller?" Several students said, "The perimeter would be smaller because it would have fewer pencils." Two students appeared to understand this concept. They said, "The perimeter of my dad's shoe would always be the same, but the nonstandard unit would change . . . paper ants are shorter than pencils."

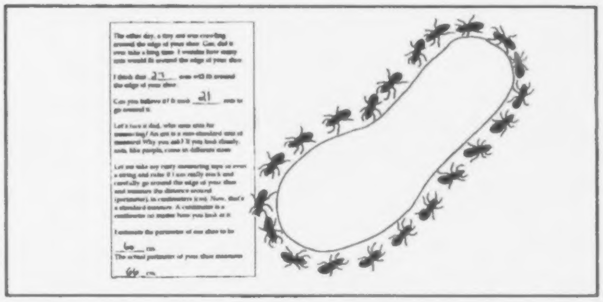


Fig. 31.5. Ant cutouts used to measure perimeter of shoe

Many of the students remembered ways to estimate area using the centimeter grid paper to trace the soles of their dads' shoes, then count the number of squares. "This is what we always do for area," said one student, "we count squares." They counted squares, knowing that area is measured using square units. The students counted the whole squares, guessed the value of the total partial squares (e.g., "I think that those fourteen pieces make up five whole squares"), or meticulously counted half and quarter squares (see fig. 31.6).

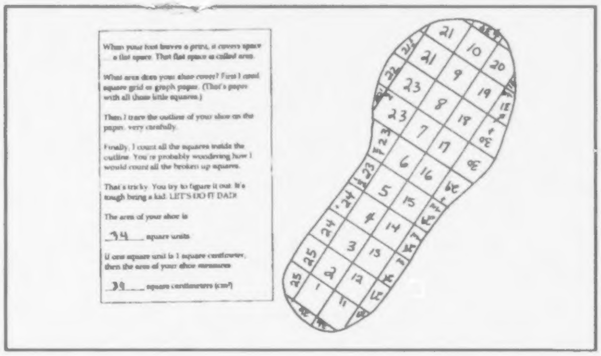


Fig. 31.6. Grid paper used to measure area of shoe

The second-grade students were eager to experiment with the triple-beam balance. As they inspected the machine, they discussed ways in which it was similar to their home bathroom scales. Their comments included, "If you push down on it, it moves"; "It has weight lines"; "You have to make sure that it starts at the center line, otherwise you might be cheating the weight." Most students thought that the triple-beam balance measured the "weight" of objects; however, through discussion, they became more familiar with the appropriate term, *mass*.

These investigations in measurement offered students opportunities to develop their understanding of measurement tools and concepts. Moreover, the students were asked to explain and justify their choices and uses of the different measurement tools. I asked some students, "Why are you using the measuring tape to measure length? Why don't you use the string?" Some students said, "Because that is what I always use," whereas others explained, "If I use the string, then I have to also use the ruler because the string doesn't have centimeters on it and the ruler is stiff." Another student said that he was "copying" his friend, yet realized, "The measuring tape has math on it . . . you know . . . a number line!"

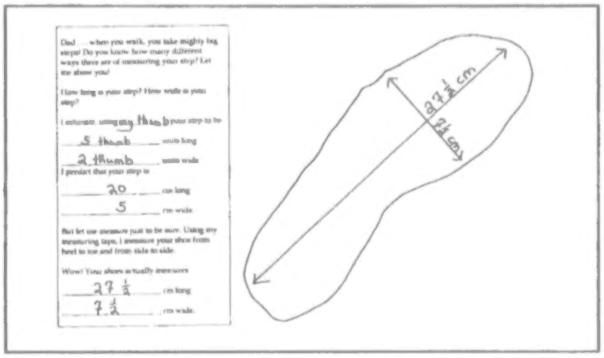
Observing and interviewing students as they use mathematical ideas to solve problems helps them to "reflect on and clarify their thinking about mathematical ideas and situations" (NCTM 1989, p. 26). As I asked questions, the students often added explanatory details about their measurements or discussed other mathematical ideas. The seventh- and eighth-grade facilitators also noticed similar reactions from other students. One facilitator stated, "The kids seemed more into it when I showed I was interested in their measuring, especially when I asked them questions."

During questioning, I could also distinguish which students understood measurement as a procedure (e.g., "You put the ruler here and get the centimeters") and which students understood the measurement conceptually (e.g., "It doesn't matter if you use a tape measure, string, a ruler, or the ants—to get the real measurement you have to use a line and the centimeter ruler of some kind"). Thus, through interviewing, we gathered assessment data about students' knowledge of measurement concepts and their skills in applying measurement procedures. Although they demonstrated a range of mathematical knowledge and skill, their interactions with one another, their teachers, and the student facilitators enabled them to build on their measurement abilities by reflecting on, and clarifying their thinking about, mathematical ideas in the context of this shoe investigation.

"Dad, You Take Mighty Big Steps!": Using a Record Sheet

The teachers and I created a series of record sheets that included pictorial and written descriptions about the shoes. The written descriptions included first-person, teacher-generated narrative text and blank spaces for students to record and describe their estimated and precise measurements using words, numbers, and pictures (see figs. 31.5, 31.6, and 31.7). All the record sheets would be compiled into a booklet called "How Do You Measure a Dad?" and given as Father's Day gifts.

As we designed these record sheets, I wondered how the students would use them. I saw and heard students read the text of the record sheet aloud over and over again, refining the quality of their oral reading skills, including intonation, pronunciation, and fluency; trying to make sense of the written words; and deciding which measurement tool to use in conjunction with the meaning of the text. One student explained, "I'm trying to read this so I can hear and know what to do." Another student said, "I'm pretending that I am talking to my dad as I am working." In fact, students explained that the story told them why they should be measuring the shoes using a certain type of measurement. "If my story part says [she reads from the text of the record sheet], 'How long is your step?' I don't put in my area measurement, right? I put my length measurement there. If I get them mixed up, then my dad will think that I don't know which is which. I know my math." (See fig. 31.7.)



Dad . . . when you walk, you take mighty big steps! Do you know how many different ways there are of measuring your step? Let me show you!

How long is your step? How wide is your step?

I estimate, using my thumb your step to be

5 thumb units long

2 thumb units wide

I predict that your step is

20 cm long

5 cm wide

But let me measure just to be sure. Using my measuring tape, I measure your shoe from heel to toe and from side to side.

Wow! Your shoe actually measures

27 1/2 cm long

7 1/2 cm wide

Fig. 31.7. Measuring the length and width of the shoe

How Does Your Dad Measure in the Class?

To make class shoe graphs, the students also recorded their measurements on data squares. These data squares were color coded (i.e., yellow for area, green for volume, blue for width, white for length, pink for perimeter, orange for mass) so that the measurement data could easily be distinguished and organized into different graphic representations, including vertical and horizontal bars, number-line plots, pie graphs, pictographs, and tree diagrams.

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The students interpreted the shoe graphs orally. Both teachers noted that the students described a range of ideas that included naming the highest and lowest data values and explaining these values. For example, the students noted the large number of long shoes. Some students thought that if the dads' feet were long, then the dads would be tall, meaning that a lot of tall dads must be in the class.

Reflecting and Rethinking

The shoe investigation enabled us to think about how the design and implementation of this series of activities influenced the process and product of students' learning. This investigation also provided authentic learning contexts for students to "acquire confidence in using mathematics meaningfully" (NCTM 1989, p. 23). All the students were eager to share the details of their mathematics learning when they presented the gifts to their dads on Father's Day. As the students used the record sheet, they rethought their mathematical ideas and rehearsed explanations about the measurements out loud. One student said, "I can tell my dad how I counted the squares to find the area of his shoe. He needs to know that it look a lot of time to count the one-half and one-fourth squares. [She points at the text of the record sheet about area. See fig. 31.6.] I wanted to make sure it was right, just in case he checked."

This shoe investigation supported all the students in thinking both mathematically and narratively about their learning experiences (Bruner 1985; Gardner 1993; Robinson and Hawpe 1986). In the process of preparing and sharing the Father's Day gifts, the students used both mathematical and narrative thinking by listening to one another, discussing ideas, and raising questions about their analyses of the attributes of shoes (i.e., sorting and classifying the shoes; estimating and measuring the shoes using different materials; collecting, organizing, representing, and interpreting the shoe data).

Bruner (1985) and Robinson and Hawpe (1986) suggest that narrative thinking helps the learner organize and make sense of experience in ways that are different from logical-mathematical thinking. In fact, narrative thinking "involves the projection of story form onto some experience or event. This may occur as the experience is taking place, in reflecting upon the experience, or in recounting the experience at a later time" (Robinson and Hawpe 1986, 115). The process is much like answering a series of questions about a story, such as what happened, to whom, and why? Answering these questions enabled students to examine the shoes and identify relevant mathematical ideas as part of constructing their stories. In fact, students used the text in several ways. Many students read aloud from the record sheet to help them take measurements from their dads' shoes. Some students envisioned the text as a conversation with their dads, and others read the text to guide them through their investigations or to help them understand the mathematical ideas. I think that students who were using their mathematical and narrative modes of thinking were developing in-depth understanding about mathematical concepts, procedures, and terminology from the strands of data management, measurement, number sense and numeracy, and patterning.

Finally, the picture book furnished a setting and context for making real-life links to mathematical concepts. In fact, the book *Shoes, Shoes, Shoes* and the dads' shoes provided real-life data for the students to analyze, sort, and classify. The narrative aspects of the activity, including the story, class discussions, and the record sheet, furnished a context in which the students analyzed and described their dads' shoes using mathematical ideas. The discussions and questioning about the use of mathematical ideas enabled the students continuously to interpret phenomena, reflect on their learning experiences, and retell the experiences.

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ACTIVITIES

Geometry and Measurement

Wheatley and Reynolds: " 'Image Maker' Developing Spatial Sense"

Refer to the list of Expectations for the Geometry Standard for grades 3–5 on page 164 of *Principles and Standards*. Identify at least five of these Expectations and, using examples from the article, explain how they are met by the "image maker" activities.

Fox: "Implications of Research in Children's Understanding of Geometry"

1. Make an enlarged photocopy of figure 26.1, and number each shape for identification purposes.
    - *a*) Carry out the Polygon Sort task. List by number the shapes sorted into each of your groups, and give the rationale for your sort.
    - *b*) Sort the polygons into subclasses, and match the Property and Name Cards in figure 26.2 with your subgroups.
    - *c*) Discuss how this activity did or did not help you deepen your understanding of the three types of reasoning described in the article.

2. *a*) Create a concept card for each of the following: trapezoid, quadrilateral, right triangle, angle
    - *b*) *Group Activity:* In your group, compare all concept cards for the same shape. Develop a list of criteria for creating an effective concept card. Discuss how these criteria help the Concept Card task elicit all or some of the three types of reasoning described in the article. Be specific.
    - *c*) *Note:* the information from the Concept Card tasks is used in the next activity.

Keiser: "The Role of Definition"

1. *a*) Read the article by Fox, and complete activity 2 before reading the Keiser article.
    - *b*) Discuss how your concept card for "angle" does or does not reflect the concept of angle developed by Keiser's article.
    - *c*) Create a new concept card for "angle" that uses the three aspects of angle mentioned in the article.
    - *d*) Write a short paragraph, with illustrations, describing as many properties of "angle" as you can.

2. Tall and Vinner (1981) make a distinction between a *concept image* and a concept definition. A *concept image* consists of all the mental pictures, properties, and processes that an individual associates with a particular concept. In contrast, a definition is a precise, unambiguous statement that explains the concept in terms of previously defined terms or more primitive concepts.

    - *a*) Discuss how the definition given in the article does or does not fit your concept image of "angle."
    - *b*) What role should definition play in teacing and learning geometry in grades K–8?

3.

DATA ANALYSIS AND PROBABILITY

INSTRUCTIONAL programs from prekindergarten through grade 12 should enable all students to—

- *formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them;*
- *select and use appropriate statistical methods to analyze data;*
- *develop and evaluate inferences and predictions that are based on data;*
- *understand and apply basic concepts of probability.*

The three articles illustrate how activities in this Standard can be incorporated across grades. Informal experiences with generating questions and collecting data in the primary grades pave the way for later in-depth explorations at the middle school level. Computer technology further enhances learning by displaying data in ways that facilitate the study of fundamental statistical ideas.

The article by Basile describes a way to introduce kindergarten children to important notions of data collection. Children generate questions about the outdoors and decide on the information that they need to collect to answer these questions. This activity provides students opportunities to refine questions, classify attributes, record data, make comparisons, and identify patterns. In addition, the activity makes connections between out-of school and in-school mathematical experiences.

Scavo and Petraroja present the results of a fifth-grade statistics project that investigated and compared the sizes of fifth- and sixth-grade classrooms in their school. The students measured the dimensions of each classroom, calculated floor-space area, and explored various ways to graphically represent the data to help them understand their results. The project culminated with a presentation to the school principal about student classroom density by grade level.

McClain presents an analysis of a group of seventh-grade students' reasonings about the use of the mean to represent a set of data. She discusses the effect of the variability in the data on the students' understandings of the meaning of the mean. Examples of students' graphical representations and their comments about the data sets illustrate the complexity of the statistical reasoning that was required to make sense of the data, graphs, and representative values. The author stresses the importance for her of understanding the students' perceptions before she designed future instructional tasks.

Collecting Data Outdoors: Making Connections to the Real World

by Carole G. Basile

THE MOST enjoyable experiences that I have had with young children are those that occur outdoors. Taking children on walks in the woods, at our local park, or simply around the school yard can prompt many discoveries about the natural world. As we walk, children gain knowledge and skills by using their senses to collect information about the world around them. Traditionally, we think of providing these experiences as part of children's scientific learning. However, direct observation is also an important piece of mathematical learning that is essential for identifying patterns, promoting problem solving, and developing spatial sense and reasoning.

Why Data Collection?

How can observations be used to learn mathematics? One way is through collecting data outdoors. Data collection is a process in which children can make connections between mathematical concepts, in a variety of content areas, and in the real world. In addition, it is possible to embellish this process with children's natural curiosity about the outdoors by offering experiences that are challenging and engaging and that support learning.

When working with children outdoors, it is interesting to note their perceptions about the world around them. Often these perceptions are misconceptions. Children will ask such questions as, "Why are all birds black?" and inquire, "What are those rats with the fuzzy tails?" when they see squirrels. Children commonly make such comments as, "Trees are only alive when the wind blows, because then they move" or "We should clean up the forest and get rid of all these leaves."

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Simply telling children that their perceptions are incorrect will not change these misconceptions. However, when children become data collectors, they look for patterns and develop reasoning skills that allow them to draw conclusions on the basis of information that they have not only collected but also observed with their own eyes. If they observe and collect information about where they find squirrels (e.g., on the ground, in a tree, on the fence), they learn about the behavior of squirrels. If children begin to observe and record the animals living in leaf litter, they soon realize the diversity of life that uses the "messy" forest floor as its home.

As children collect data, they see how prior knowledge about number, size, shape, and pattern relates to things around them. For example, they may use numbers to help them count and record the number of petals found on different flowers, they may use nonstandard measurement devices to examine the circumference of trees and compare sizes, or they may simply collect data about all the outdoor things they find that have ABAB patterns. Data collection builds beliefs about what mathematics is and about what it means to know and do mathematics, and it supports children's view that they are mathematicians. Children who collect data can portray themselves as mathematicians and begin to see how mathematics can be used to solve real problems.

In the Classroom

A group of kindergarten students began their investigation of data with the following question from the teacher: "One thing mathematicians do is collect data. Does anyone know what data can be?" No one knew. One boy asked if it was like collecting bugs. The teacher wrote "data" on the chalkboard and then walked around the room pointing out data that the children had already collected, like the information on their weather chart, children's birthdays, and their favorite foods. Then she explained that collecting information, or data, helps you to learn something that you might not have known before.

The teacher then asked the children if they would like to know anything about the animals in their school yard that they did not know before. Their responses included how caterpillars grow, how big is an anthill, how much do animals weigh, how big are the animals, what animals can we see, and what animals can we hear. The teacher wrote all the ideas on chart paper, and as a group, the children decided which idea to investigate. They decided that they would use their senses to see which animals they could see and hear.

Once the children had made this decision, the teacher encouraged them to think about how they would keep track of the data. They decided to use their journals. One child suggested writing everything down in a list; another said that she could remember it in her head; and a third suggested using two pages, one for what they would see and one for what they would hear. In the end, the third idea was modified; they chose to draw a line down the middle of their papers and write or draw the animals that they could see on one side and those that they could hear on the other side. At the top of one column, they drew an ear; at the top of the other column, they drew an eye (see fig. 32.1).

Deciding what kind of data to collect and how to record the information can be difficult for inexperienced children. The more often that students get an opportunity to ask the questions themselves, the better they become at recognizing data that they want to collect in the future. NCTM (1992) states that as students are involved in generating questions and collecting data, they will be able to make more sense of data.

Sorting Data

Sorting is an excellent way to encourage children to think about important features of data that lead to classifications that make sense. Two important questions concern sorting data: (1) Do we think about sorting data before or after they are collected? and (2) Which sort gives us the best answer to our original research question?

In the example described previously, the kindergarten children decided to sort their data before collection into "animals they saw" and "animals they heard." These categories helped them think about the questions that they wanted to answer and how to organize their record keeping of the collection. When they returned to the classroom, however, they realized that the data could have been sorted into "animals they saw," "animals they heard," and "animals they saw and heard." This recategorization of the data was valuable because it helped the children understand that classification systems can grow and evolve as data are collected.

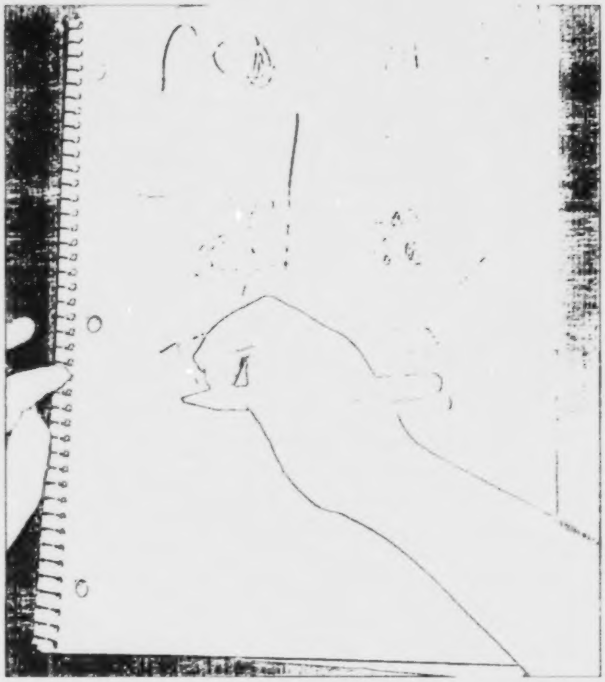

Fig. 32.1. The "ear" and "eye" columns helped organize the data.

It is also possible to collect data as separate pieces of information, without any categorization decided on beforehand. Children could be given small sheets of paper stapled at the top, and each time they use their senses to observe an animal, they could draw a picture or write the name. When they return to the classroom, they could pull their "books" apart and decide how to sort the data into groups according to a common characteristic. This more open-ended approach would also be beneficial in helping children understand the various ways that data can be categorized. It would also help them think carefully about the question they are researching.

For example, another class decided to measure and record the size of the animals that they saw. The class had been learning nonstandard measurement and saw this task as being an opportunity to use their new skill outside. Many suggestions were made about what they could use for a measuring device, but they finally decided on Unifix cubes. Before they collected the data, the children talked about data-collection procedures and identified those that would not harm any of the animals that they were measuring. The following recommendations were listed on the chalkboard: "Don't touch the animal with your hands or with the Unifix cubes"; "If an animal is too far to reach, just estimate"; "If an animal might hurt you (like a bumble bee), just estimate"; and "We all do not have to measure the same animal. One person in the group can measure while the rest record the information." (See fig. 32.2.) Once the procedures were agreed on, the children decided to collect the data by designing a chart in their notebooks and by recording a picture of the animal and the number of cubes needed to measure it.

In this example, the children's data-collection charts were not organized in any particular way. Therefore, the children were placed in small groups and asked to think about how they might sort their data to represent them to others. One group sorted by the type of animal they saw (mammals, reptiles, insects, and so on). They had forgotten about the original purpose for collecting the data. The teacher then referred them to their initial question and asked, "Does this grouping help us answer what size the animals are that we found outside?" At the end of the lesson, each group presented its groupings, and the class members discussed which sorting method they thought best answered their research question.

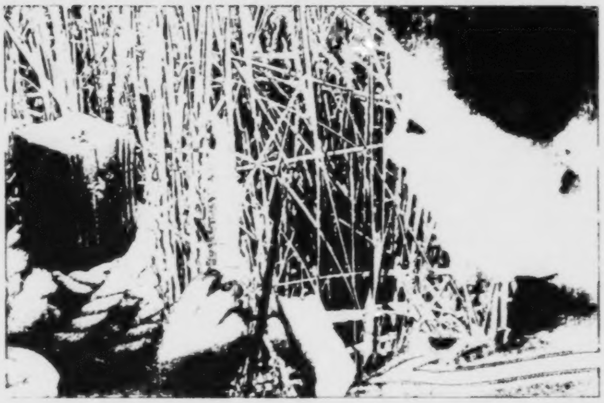

Fig. 32.2. "This is a caterpillar, I think."

Representing Data

When the students returned to the kindergarten classroom with their journals, the teacher posed the following problem:

> I have a friend who studies animals and would like to know what we found in our school yard. We have collected a lot of data. How can we display it on one piece of paper so that she might be able to see what we found?

Thinking about Classification

Giving children the opportunity to think about whether their classification helps them answer their research questions is essential for developing a deeper understanding of data analysis. Encouraging children to create different ways of sorting data helps them recognize the variety of possibilities for collecting and categorizing data. One can find numerous examples of scientists and social scientists who must think about how to sort the data that they want to collect before and after collection. For example, public relations firms that develop questionnaires, marketing assessments, and opinion polls must think about the information that they want before they design an instrument to collect the data. In a survey, they might use such demographic information as age range, salary range, and level of education. The questions might also be categorized by such factors as knowledge, preferences, and practices. In contrast, social scientists who collect qualitative data might organize or categorize their data *after* they have collected it. Researchers might want to know people's opinions about a particular program and then ask an open-ended question, such as, "What do you see as the benefits of this program?" They would then sort, categorize, and quantify the responses after they were collected.

The children said that they could write small and that each person could write one thing that they found. When the teacher demonstrated to them that some of them saw the same things while others saw or heard different things, she asked whether they could find a way to get all the information on the chart without repeating any information. One student said, "Make a chart like our journals." The student was then asked to show the group what he meant. He drew a line down the middle of the chart and drew an eye and an ear at the top, just like in the students' journals. Once the data from the entire class were compiled on the chart, the students discussed what the graph illustrated. They talked about how many animals they saw and heard and why more animals were in one category than in another.

To help the students think about other ways to represent their data, the teacher gave students Unifix cubes and asked them to choose two colors, using one color to represent the animals that they heard and the other to represent the animals that they saw. A student's representation is shown in figure 32.3.

Davis and Maher (1997) explain that everyday experiences can provide children with the building blocks for learning mathematical ideas. Further, when children can represent these ideas—whether on paper, with manipulatives, or mentally—learning is extended. Therefore, after data are collected, children need to explore ways to represent those data. The events described here show that children can think in various ways to organize the data that they have collected. This task of organizing data allows children the opportunity to sort and classify real data for a real purpose.

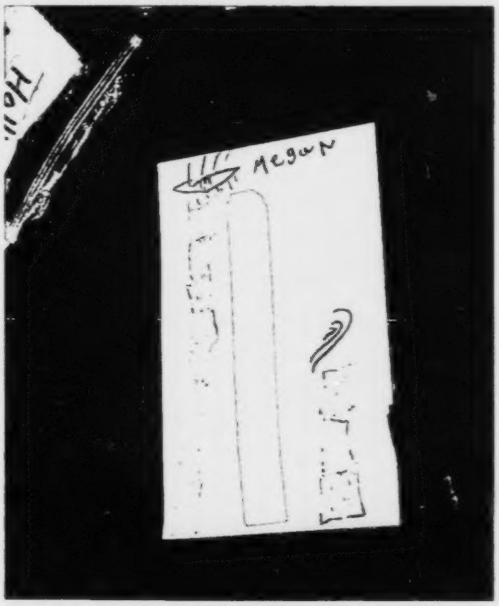

Fig. 32.3. Blue and red cubes represent data in the "ears" and "eyes" column.

Asking Questions

The questions that the teacher asks about data throughout the process of formulating a question, collecting data, representing data, and interpreting data are crucial. They help children think about the important mathematical understanding involved in data collection, as well as clarify their own perceptions of the natural world.

The following list represents the kind of questions that the kindergarten teacher asked that fostered mathematical thinking. Her questions helped students—

- formulate research questions: "What kinds of animals do you think we could find in our school yard? What do you want to know about them?"
- think about sorting by important features and record keeping: "How can we keep track of what we see and what we hear without getting them mixed up?" "How will we remember what we saw and what we heard?"
- think about reclassifying: "What else is the same or different about some of the animals we saw? If we were to make a new chart with only the animals we saw, and we had to put some in one column and some in the other column, how could we do that so that all the animals in one column would have something the same?"
- represent the data: "If we wanted to draw a picture of our data to show another class what we found, how could we do that?"
- describe what the representation tells us and why we think it has occurred: "Which side has more animals? Why do you think we saw more animals than we heard? If we collected these data at night, what do you think the results would be?"
- clarify ideas about the natural world: "Which of these animals are insects? How do you know? Why do you think we found more insects than any other animal? Why do you think we did not see any frogs? Where could we find more information about the songs of birds and frogs?"

Using the Data Later

One rule is important about collecting and compiling data: never throw anything out. The information collected by your students can be used again to compare data sets and see change over time. By documenting such variables as weather conditions, time of day, study area, or date, children can observe differences in data sets from one day to the next or from one area of the school to another. For example, one teacher placed bird feeders in the back of the school and none in the front of the school. The children decided to collect data to see whether they saw more birds in the front or the back. They recorded the data by using pictographs for each location. For the data collection in front of the school, the children decided on the following four categories: in the air, high in the trees, low in the bushes, and on the ground. For the back of the school, the students decided that they should add one more category: on the feeders.

By displaying one data set next to another data set, the children were able to make comparisons and see similarities and differences between the two sets of data. They offered the following observations: "There are more birds in the back of the school than in the front," "There are more birds at the birdfeeders than any location in the front of the school," and "There are more birds on the ground in the back of the school." When they were asked to think about why these differences existed, the children excitedly offered possible explanations, such as, "The birds are hungry; that's why we see so many at the birdfeeder" and "There are a lot of birds on the ground in the back of the school, because a lot of the seeds have fallen to the ground from the birdfeeder!" Making comparisons among data sets prepares young children to think like statisticians as they discuss possible interpretations for the similarities and differences that have occurred.

Conclusion

The process of data collection gives young children a wealth of opportunity to think like mathematicians and apply their reasoning about data to the real world. It also introduces children to ideas of more abstract mathematics as they sort and classify data, create graphs, compare data sets, examine patterns, and interpret graphical representations.

It is crucial that educators create meaningful mathematical environments by providing real-life contexts and setting the foundation for more abstract mathematics. Both the National Council of Teachers of Mathematics and the National Association for the Education of Young Children call for young children to learn in realistic contexts and to study the world in which they live. Taking children outdoors gives them real experiences that they might otherwise not have had, especially those in urban areas where green space is limited or unsafe. The more we can integrate real-life activities into classroom learning, the more students will be able to recognize the importance of what they are learning.

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Adventures in Statistics

by Thomas R. Scavo and Byron Petraroja

WE HAD just returned from the science fair, and the scientific method was fresh in the students' minds. In this context we introduced the following unit on data analysis, which we hoped would not only satisfy the district's elementary-statistics requirement but also involve the students in a long-term mathematics project in a meaningful and relevant way.

Our fifth-grade class was perhaps typical in many respects: a culturally diverse group of twenty-four children sharing a medium-sized classroom in an inner-city-neighborhood school. On more than one occasion, we heard comments about the lack of space in our classroom and that other classrooms in the school seemed to have more space. We thought about these comments and how we might capitalize on the students' natural curiosity. We began with the following problem statement:

> **Problem:** Are the areas of the sixth-grade classrooms larger, on the average, than the areas of fifth-grade classrooms?

We then asked the students to hypothesize the answer to this question and to document their responses.

During the initial stages of problem development, a discussion occurred concerning the practical applications of our ultimate findings. A variety of possibilities surfaced, most notably the fact that the principal could use the information when setting up the classrooms for the next year.

Measurement

In anticipation of the primary measurement task described here, we talked about the length and width of our classroom and how we might go about measuring it. We estimated these measures by sight and wrote down our estimates for future reference. We also discussed what we meant by the area of the room, which we then computed using our estimates of length and width.

Working in pairs, everybody measured the length and width of our classroom. First, one student from each team measured while the other recorded, and then they switched roles, measuring the length and width again. The two sets of measurements, a kind of check on their work, were recorded on data sheets. When all the teams had completed the measurement task, the data were written on the chalkboard and compared. Discrepancies between the measurement pairs were noted. Those that could not be attributed to measurement or round-off error were taken again.

We arranged for the students to measure each of the fifth-grade and sixth-grade classrooms in our school, fourteen altogether. Of course, the other teachers were told in advance about our plans to enter their classrooms. So that the activity would go smoothly, parents and staff were recruited to supervise the classroom visits. We tried to have one adult for every four students.

At the designated time, the students were again paired, classrooms were assigned, and metersticks were handed out. (Metersticks were chosen as our measuring device because they were plentiful and

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Measurement Data Sheet

My name is ______________________________.

My partner's name is ______________________________.

We are measuring room # ______________________________.

Measure the length and width of the room to the nearest centimeter:

We measured the room to be

________ meters and ________ centimeters long

and ________ meters and ________ centimeters wide.

We measured the room twice. The second time it was

________ meters and ________ centimeters long

and ________ meters and ________ centimeters wide.

Do the following calculations when you get back to our room:

We calculated the room to be a total of ________ meters long and ________ meters wide.

The area of the room is ________ square meters.

Fig. 33.1. Recording sheet for the first measurement

familiar to the students.) On the data sheets (fig. 33.1), each team recorded their measurements to the nearest centimeter. As before, the students measured the rooms twice as a check on their work.

Before we could calculate the area of each classroom, the data had to be converted to common units; for this task, the meters were to be expressed in decimal form. For example, one team found its room to be seven meters and sixty-two centimeters long, but how long was the room in meters only? (7.62 m) Likewise, how wide was the room in meters? These conversions were difficult for some students, but we found that teaching the concept in the context of this unit was most effective.

After each team had determined the length and width of its assigned room in meters, members were given a calculator and asked to compute the area of the classroom. This task was easy for the majority of students. As they wrote their answers on the class data sheet, we made sure that they wrote down the units as well (e.g., 64.2 square meters). In fact, all measurements were accompanied by the appropriate units; a number by itself was not sufficient. See table 33.1.

TABLE 33.1
Area and Number of Students in the Fifth-Grade and Sixth-Grade Classrooms

Grade	Room	Area	Students
5	203	43.0 m^2	10
	204	70.0 m^2	25
	205	80.1 m^2	25
	206	64.2 m^2	23
	208	61.1 m^2	24
	209	61.8 m^2	24
6	105	60.7 m^2	24
	107	70.0 m^2	17
	108	62.3 m^2	26
	109	70.5 m^2	17
	110	64.1 m^2	26
	112	92.0 m^2	16
	207	59.0 m^2	19
	203	17.6 m^2	4

The next class period was spent examining the data. The class data sheet was hung at the front of the room, and discussion began. Which classroom has the largest area? Which room has the smallest area? We also asked these questions with respect to grade level: for example, which fifth-grade class has the largest room? We already knew some of what we discovered, but some facts, such as the size of room 112, surprised us.

Note that room 203 appears twice in table 33.1. This particular room, a mixed class of both fifth and sixth graders, presented us with special challenges and opportunities. To facilitate the discussion, we constructed another data sheet (see fig. 33.2). Our task was to conceptually allocate the total area of the room according to the relative numbers of fifth-grade and sixth-grade students. This pursuit required a knowledge of percents, and we found ourselves digging into the textbook for some help.

It did not occur to us at the time, but we could have further investigated the concept of percents by asking other questions. For instance, of all fifth-grade and sixth-grade students, what percent are fifth graders? What percent are sixth graders? What percent of all

Special Data Sheet

Student data for Mr. Walsh's room (#203)

	Number	Percent
Fifth graders:		
Sixth graders:		
Total students:		100%

Mr. Walsh's room is a total of _____ meters long and _____ meters wide.

The area of the room is __________ square meters.

The portion of the room allocated to fifth graders

= (percent of fifth graders) × (total area of the room)
= ________ × ________
= ________ square meters.

The portion of the room allocated to sixth graders

= (percent of sixth graders) × (total area of the room)
= ________ × _______
= ________ square meters.

Fig. 33.2. Data sheet for a multigrade class

students in our school are fifth graders? The latter question would have required additional information that we did not have.

Graphing

Having worked only with numerical data thus far, we sought to deepen our understanding by graphing the data. This group of fifth graders had a knack for drawing nice graphs and really enjoyed this aspect of the project. With little direction, the students set to work drawing graphs of the areas of the fifth-grade and sixth-grade classrooms. Classmates worked in pairs while drawing these graphs. We were not sure how team graphing would turn out, but we found that the children had few problems sharing tasks. In fact, we were amazed at how easy and productive it was for two children to work on one graph simultaneously.

Most of the students chose to draw a line graph, since we had drawn this type of graph earlier in the year. For the data with which we were now working, however, a bar graph seemed more appropriate, since it more clearly showed the differences among classrooms. So we asked the students to convert their line graphs to bar graphs, which they did without much trouble. The next day we put up a transparency of a carefully drawn bar graph of the areas of the fifth-grade and sixth-grade classrooms (see fig. 33.3). This aid gave us another opportunity to discuss the apparent differences among classrooms and to speculate about what might be done to compensate for these differences. Room 112, for instance, was obviously larger than the other classrooms and should be able to hold more students.

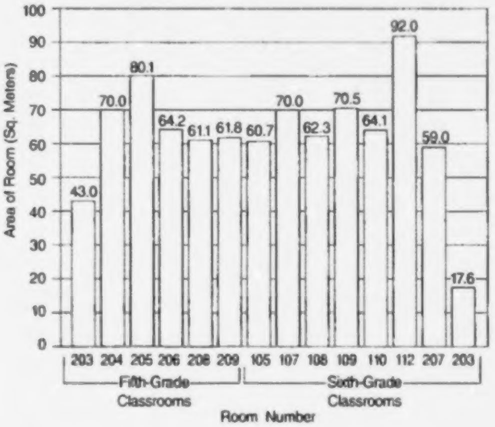

Fig. 33.3. Area in square meters of the fifth-grade and sixth-grade classrooms

The discussion led us to consider another variable, namely, the number of students in the fifth-grade and sixth-grade classrooms. We repeated the previous activities for this new variable (see fig. 33.4) and compared the area with the number of students by overlaying the two transparencies. The class found this composite view of the data especially enlightening.

Computation

The next concept involved the average area of the classrooms. In our experience, students have little difficulty computing an average, but they invariably have problems interpreting what an average represents. For our purposes, the area bar graph in figure 33.3 provided a convenient graphical aid with which to discuss averages. As a class, we did the following thought experiment: with a scissors, we cut off the tops of the

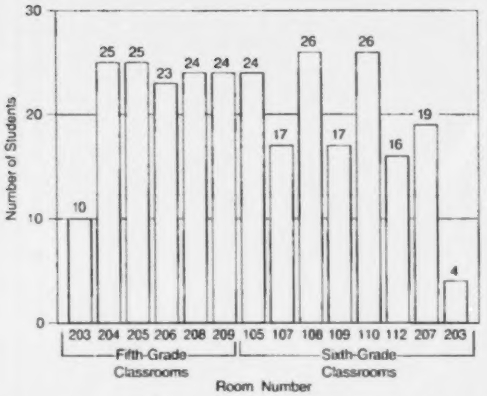

Fig. 33.4. Number of students in the fifth-grade and sixth-grade classrooms

taller bars and stacked them on top of the shorter ones so that all the bars were of the same height. This uniform height was the average area of the fifth-grade and sixth-grade classrooms. We found that the students had little difficulty performing this thought experiment, and, in fact, the class as a whole experienced an "Aha!" sensation when shown this graphical interpretation of average.

Since this visual demonstration was so effective, we chose not to make this concept more concrete. However, we could easily have done so using construction paper and scissors, and we plan on doing it next year so that tactile learners might benefit as much as visual learners.

Now that they had a better understanding of average, we asked the children to compute the average area and the average number of students in the fifth-grade and sixth-grade classrooms, which they did rather easily using calculators (see the first two rows of table 33.2). To keep the level of understanding high, and for the purposes of evaluation and assessment, we also asked students to show their work, which could be organized vertically or horizontally. For example, we wanted them to write something like

$$\frac{43.0+70.0+80.1+64.2+61.1+61.8}{6} = \frac{380.2}{6} = 63.4 \text{ m}^2,$$

$$\frac{60.7+70.0+62.3+70.5+64.1+92.0+59.0+17.6}{8} = \frac{496.2}{8} = 62.0 \text{ m}^2,$$

and compute the approximate areas with a calculator. Some students resisted writing the problem, however, arguing that the calculator made it unnecessary. We believe the opposite and insisted on seeing their work.

For homework, we asked the students to compute averages for the fifth and sixth grades combined (see the last row of table 33.2). Instead of adding all the values and dividing by 14, some of them incorrectly computed the average of the two averages. The answers obtained by the two methods did not agree, since different numbers of fifth-grade and sixth-grade classrooms were involved. Computing the average of the two averages, students found the average area of all the classrooms to be

$$\frac{63.4+62.0}{2} = 62.7 \text{ m}^2,$$

whereas the correct calculation is

$$\frac{380.2+496.2}{14} = 62.6 \text{ m}^2$$

using the sums obtained earlier. The fact that the two values turned out to be nearly the same is unfortunate, since some of the students thought it was a minor discrepancy because of round-off error. We chose not to pursue this issue with our students, but the reader may wish to explore this situation by constructing a data set in which a greater difference occurs between the two results.

TABLE 33.2
Average Area and Average Number of Students in the Fifth-Grade and Sixth-Grade Classrooms

Grade	Area	Students
5	63.4 m^2	22
6	62.0 m^2	19
Combined	62.6 m^2	20

Analysis

On the class data sheet, the teacher recorded the number of students in each classroom beside the area measurements. These data were obtained from the school office but could have been obtained directly as part of the primary measurement activity. The children soon realized that the number of students in each room was an important factor to be considered.

For instance, with twenty-four fifth graders in our room, we felt cramped at times. What if three new classmates joined our group tomorrow? Would we be more or less crowded? Everyone understood that adding students to any given classroom meant less space per person. In fact, we next wanted to compute the amount of space that each student had.

The following activity not only was fun, but it also helped the children understand what we meant by "area per student." Everybody was asked to stand and form a two-dimensional human grid throughout the classroom. We held up a geoboard to illustrate the configuration we wanted to achieve. Students spread their arms and adjusted their positions so that they were equally far from each of their neighbors; then they turned 90 degrees and adjusted their positions again. We kept turning and adjusting until the students were approximately equidistant from one another. Then we estimated the area per student by sight and recorded our estimates.

Next we actually computed the area per student in our own classroom. We had already measured the area of our room, and, of course, everybody knew how many children were members of our class. Now what should we do with these two numbers? Some students realized immediately that they should divide area by the number of students, but others did not. It is important at this stage to give the students enough time to realize this fact on their own. In our view, it is something that the students should discover for themselves.

Using calculators, each student computed the area per student in our classroom and recorded their answers on data sheets. Once again we emphasized the importance of also recording the units, that is, square meters per student, which we abbreviated to "m^{2}ps"; we considered the alternative notation "m^2/s" but in the end adopted "m^{2}ps" because it resembles the familiar notation "mph" for miles per hour. Finally, we compared our computed answers with the earlier estimates.

We were now ready to compute the area per student in the remaining fifth-grade and sixth-grade classrooms. The students paired off, and each team was asked to compute the area per student using the measurement data obtained earlier. As each team completed its calculation, a representative recorded the results on the class data sheet posted at the front of the room (see table 33.3). Some students began to see differences in the data in the area per student among various fifth-grade and sixth-grade classrooms. The bar graph in figure 33.5 helped them to visualize these differences.

We then asked the children to make the following additional area calculations:

TABLE 33.3
Area per Student in the Fifth-Grade and Sixth-Grade Classrooms

Grade	Room	Area	Students	Area per Student
5	203	43.0 m^2	10	4.3 m^2ps
	204	70.0 m^2	25	2.8 m^2ps
	205	80.1 m^2	25	3.2 m^2ps
	206	64.2 m^2	23	2.8 m^2ps
	208	61.1 m^2	24	2.5 m^2ps
	209	61.8 m^2	24	2.6 m^2ps
6	105	60.7 m^2	24	2.5 m^2ps
	107	70.0 m^2	17	4.1 m^2ps
	108	62.3 m^2	26	2.4 m^2ps
	109	70.5 m^2	17	4.1 m^2ps
	110	64.1 m^2	26	2.5 m^2ps
	112	92.0 m^2	16	5.8 m^2ps
	207	59.0 m^2	19	3.1 m^2ps
	203	17.6 m^2	4	4.4 m^2ps

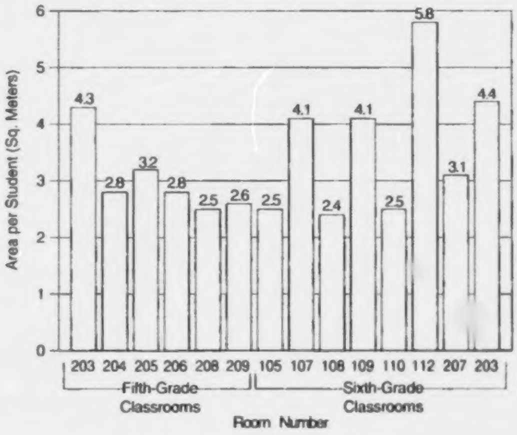

Fig. 33.5. Area per student in the fifth-grade and sixth-grade classrooms

1. What is the average area per fifth-grade student? (2.9 m^{2}ps)
2. What is the average area per sixth-grade student? (3.3 m^{2}ps)

203

3. What is the average area per student for the fifth and sixth grades combined? (3.1 m^2ps)

These calculations gave us considerable difficulty. For instance, to compute the average area per fifth-grade student, instead of computing

$$\frac{43.0+70.0+80.1+64.2+61.1+61.8}{10+25+25+23+24+24} \approx 2.9\ \mathrm{m}^2\mathrm{ps},$$

most of the students calculated

$$\frac{4.3+2.8+3.2+2.8+2.5+2.6}{6} = 3.0\ \mathrm{m}^2\mathrm{ps},$$

which is incorrect in this situation because different numbers of students were in the six classrooms. Unfortunately, the two answers are nearly equal, which made it difficult for us to convince the students that one is correct and the other is wrong.

We also asked the following challenge question:

> The principal is thinking of converting the teachers' lounge into a classroom. How many students should she assign to this new classroom?

Some of the students answered the question with another question, namely, what is the area of the teachers' lounge? This response is exactly what we wanted, since it indicated that the students could apply what they had learned to new situations.

The average calculations confirmed what we had observed in figure 33.5: sixth graders have more space, on the average, than fifth graders. We discussed this result as a class, and again someone singled out room 112 as being very different. We wondered what would happen if room 112 were out of the picture altogether. Would the fifth-grade and sixth-grade classrooms be more equitable in terms of area per student? This question led us to look at the data in different ways and to consider other numerical measures that are less sensitive to outliers. One such statistic is the median.

During the next class period, we proceeded to rank the area-per-student values from smallest to largest. From this ranking, the students easily identified the range of the data, but our immediate goal was to calculate the median. We decided to work initially with contrived data sets of various sizes, first an ordered set of five numbers and then a set of six numbers. When asked to give the median of the latter data set, some students immediately and correctly suggested a value midway between the third and fourth numbers, but not everybody understood what we were doing. In fact, we had a difficult time explaining why one might want to "compute" a median in the first place.

We had hoped that looking at other measures of central tendency, that is, median and mode, would help to strengthen the students' understanding of the data. Unfortunately, this was not the case. The students simply could not understand why we were sorting the numbers and picking out the middle one. For most of them, the average seemed good enough.

Presentation

As the project developed and the usefulness of the data became apparent, we decided to invite the principal into our classroom to share the results. We thought that perhaps she would find the data useful when it came time to assign students to classrooms the next year. We divided the project into parts and assigned three or four students to each part, asking them to study the data and the charts and to prepare a short presentation. Each group was responsible for a single aspect of the project. For example, one group of students introduced the area-measurement chart (table 33.1), and two other groups presented the corresponding bar graphs (figs. 33.3 and 33.4). We formed a group for each chart and graph and another to summarize and interpret the data at the end of the presentation. Realizing the time constraints involved and wanting the presentation to go as smoothly as possible, we rehearsed the entire presentation twice, on different days.

Excitement was in the air the day the principal was to arrive. We were pleasantly surprised when she brought the district's superintendent of instruction with her. Everyone was a little nervous, and some were timid, but our anxiety wore off and the presentation commenced. The principal and superintendent asked some very pointed questions that the students handled quite well. At the end of the presentation, the principal told the children that the data they had collected were indeed useful to her. Because of this outcome, the students were justifiably proud of their work.

Extensions

This unit could be extended or enhanced in several ways:

- The presentation could be given to other school groups, such as the parent-teacher organization or the student council.
- We could determine maximum occupancy rates for our building, from the viewpoints of both the school district and the local fire department. It would be interesting to see what kind of formula the fire marshal uses to arrive at these occupancy rates. Perhaps the size of hallways is an important part of this determination.
- Other classes in our school might want to do similar projects at other grade levels. One would expect to see significant differences in area per student in the primary grades, for instance. It would be both intriguing and instructive to compare the results obtained across grade levels and find out why these differences exist. The resulting data would be even more useful to the principal as she makes her decisions about class assignments for the coming year.
- In the future, we hope to conduct this project in parallel with fifth-grade classes in other schools in our district. Then we could make a field trip to the other school, have lunch, and compare notes. Anything we can do to get our students talking about mathematics outside the classroom will be beneficial.
- A joint project with other schools could be conducted using the Internet. One thing we could do is measure our respective playgrounds, for example, using a trundle wheel; communicate our results via electronic mail; and publish the summary data on appropriate Usenet newsgroups, like k12.ed .math. The possibility of collaborating with a group of students across the country or around the globe is an exciting prospect.

Conclusion

Our goal was to engage the students in a meaningful application of mathematics. We chose a project format requiring a significant amount of sustained work, hoping that the excitement and enthusiasm generated would carry over into the mathematics itself. We tried to make the project fun for the students, but at the same time we were careful about our work and how we presented ourselves to the rest of the school community. The students walked away from this unit feeling good about themselves and what they had done.

The next time we undertake such a project, we will have the students keep a portfolio. This approach will help us document our work and allow us to refer to earlier results. With a portfolio, for example, it would be easy to compare our initial estimates with the actual measurements, something we want to be more careful about next time.

We spent more time on the project than we had originally planned, mainly because it kept pulling us in different directions that required certain mathematical skills that we had not anticipated. In any event, it was time well spent, and we hope to do it again.

Reflecting on Students' Understanding of Data

by Kay McClain

ONGOING discussions about students' experiences with data analysis include debates about the role of graphical representations in supporting students' understandings (Shaughnessy 1992; Lehrer and Romberg 1996; Hancock, Kaput, and Goldsmith 1992). More specifically, do students first need to know how to construct various types of graphs before they can engage in an analysis of data, or can they learn how to construct various types of graphs by engaging in data analysis? Further, can they engage in data analysis before they have acquired a conceptual understanding of the multiple forms of data representation? These discussions also include a debate over the role of representative values, such as the mean, mode, and median (Mokros and Russell 1995). What are students' understandings of representative values? How should representative values be introduced? What role does students' understanding of representative values play in their ability to analyze sets of data?

This article is intended to address these issues by discussing what happened as I observed several groups of seventh graders working on performance tasks designed to assess their understanding of both graphical representations and representative values. During my interactions with these students, their activity made me reflect on my then-current beliefs about what it means to know and do "data analysis" in the middle grades. In particular, as I observed students attempting to find a way to "represent" a data set by debating the advantages and disadvantages of the graph as a visual, single-glance impression of the entire data set versus the mean as a single-number numerical summary, I began to rethink what might be involved in an instructional sequence that addresses these concepts. By taking an in-depth look at the way that students reasoned about representing a set of data and conveying my reflections on their processes, this article highlights the importance of teachers' supporting students' development of conceptual understandings of multiple forms of data representation and representative values in the context of ongoing data analysis.

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Classroom Episode

A shift in the way I thought about students' experiences with data occurred as I was working with seventh-grade students on several performance-assessment tasks. The tasks were intended to yield information about how the students reasoned about a set of data and were designed to be accomplished in group settings. Specifically, I wanted to try to understand how the students might go about organizing and subsequently representing a set of data to generate a summary of the information. This preassessment information would, in turn, be used to guide the research team's decisions about instructional tasks that would be introduced in a unit on exploratory data analysis. As a result of the need to clearly understand students' interpretation of, and reasoning about, the task, two of the group discussions and the subsequent whole-class discussion were videotaped. Viewing the tape gave me the opportunity to analyze the students' ways of reasoning beyond what I observed as I monitored the different groups.

One task asked students to summarize the results of a hypothetical survey to create a report for the principal and parents. The survey results included the number of hours of television that thirty seventh graders watched in one week. The task is shown in figure 34.1. In anticipating how the students might begin to reason about organizing the data, it seemed obvious to me that a histogram would be a clear and concise way to represent the information. For me, the histogram would preserve the variation in the data set while simultaneously giving a holistic impression of the trends and patterns within the data set without the need to digest each piece. The use of the mean for this particular set of data would appear to oversimplify the information, eliminating the nature of the variation. However, as this task was designed for preassessment purposes, I was not sure how ready-to-hand the use of histograms was for the students. Further, my assessment of their performance would not be based solely on whether they made a histogram and made it correctly but would focus more on how they reasoned about organizing and representing the data. My colleagues and I needed this type of information to guide the design of instructional tasks for the unit on exploratory data analysis.

How Much Television?

Below are the results of a survey taken of 30 seventh graders to find out how many hours of television they watch in a week. The principal has asked you to summarize and represent these data in some form so that parents will be able to understand them quickly when they are posted on the bulletin board. The principal also asks you to write a short report for parents, explaining what the data show.

1.5	21	12.5
0	2.5	15
23	19	4
14	8	16
13.5	16.5	6
4.5	9	18
5	10.5	8.5
6	3	9
11.5	3.5	19.5
13	10	9

Fig. 34.1. Survey data given to students

As students began working in their groups, I walked around the classroom to try to monitor their activity and begin to understand how they were reasoning. Several groups began by finding the mean of the data set. One group asked for the calculators, and it could be argued that their request triggered a request from the other groups. Nonetheless, several groups began to methodically calculate the mean of the thirty responses.

As I continued monitoring the groups, I noticed that of the groups that found the mean, some of them subsequently rejected it as inappropriate. They reasoned that just reporting a mean of 10.56 for this set of data was insufficient, arguing that just one number did not provide enough information for this particular data set because "a bunch of the numbers were way above and a bunch of the numbers were way below" the mean. They then proceeded to discuss how to make a graph that would better represent the data, keeping all the features visible. However, for one group, the use of the mean became an intense topic of discussion. In particular, Amee and Latisha argued that you cannot use the mean, whereas Tony insisted that the assignment was to tell about the thirty students "all together," which was exactly what the mean did. Tony seemed to recognize the need to take the variation into account, and he thought that the mean, as an arithmetic average, did just that.

Amee: You cannot average it out, Tony.
Latisha: You cannot average it.
Tony: If I want to, I can.
Latisha: Listen, Tony . . . .
Tony: [Interrupts] They said all together, all together now.

Latisha and Amee then tried to explain to Tony that some of the students in the survey only watched 1.5 hours, so the average is "way off."

Amee: Tony, when you average it out, it is supposed to come somewhere close to [1.5].
Tony: It's not supposed to come out close to 1.5.
Amee: But you cannot do [the average].
Tony: I can. You really can, it just might not be accurate.
Amee: It's way off, it's very way off. It's so off, you cannot use the answer.
Tony: Yes, I can.
Amee: Well, you use the answer, but I'm not going to use that answer.

I found it interesting to note that this discussion appeared to focus on whether you could actually use the mean, not whether it would provide a clear representation in this particular task. I interpreted the students' discussion as suggesting that in certain instances you can use the mean and in others it simply does not work. However, their inability to clarify their true understandings was problematic for me. As a result, I intervened in the conversation to help clarify my understanding of their reasoning.

Teacher: Let her give her argument, and then you can give yours.
Latisha: Tony, when I averaged it, it came out to 10. This person here, this is 23 hours.
Tony: So? They said all together, all together, not just one person by hisself [sic].
Teacher: [Latisha's] agreeing with you that [10.56] is the average. What she is saying is that she doesn't think it is a good way to tell the story of these numbers because there is so much variation here. Because if you say 10, then people might think that everybody watches 10. Some don't even watch any.
Nathan: See, you don't watch 10, you watch about 20.
Tony: So, y'all are just picturing one guy by himself.
Amee: That's what you are doing.
Tony: No, I'm not.

At this point, I was still unclear whether Nathan, Latisha, and Amee thought that the variability in the data set resulted in the mean's inability to give a complete report of the situation in this particular instance or whether they thought that since some of the data points differed so markedly from the mean that therefore it did not work. I clearly needed to understand those two very different interpretations of the mean to plan how to proceed with further instructional tasks.

As a result of the group's inability to reconcile their differing interpretations, Tony decided to use the mean and others in the group decided to make a graph. They began by discussing the possibility of grouping the data. Initially they discussed grouping the data from 0 to 10, from 10 to 20, and from 20 to 30. However, Nathan pointed out that the data did not go to 30, so they changed the upper bound to 25. I found this decision intriguing, as the largest data point was not 25 but 23, and changing the upper bound from 30 to 25 would in no way affect the height of the bar. It would, however, create unequal data intervals. The fact that their intervals were not of the same size was not problematic for the students, nor was it represented in the width of the bars in their graph (see fig. 34.2 for their final version). However, since I was anticipating students' decision that this set of data could best be represented by a histogram, the issue of inconsistent intervals was problematic for me. Further, I was not clear whether the students were making a modified histogram or simply grouping the data points into categories that they named with numeric intervals.

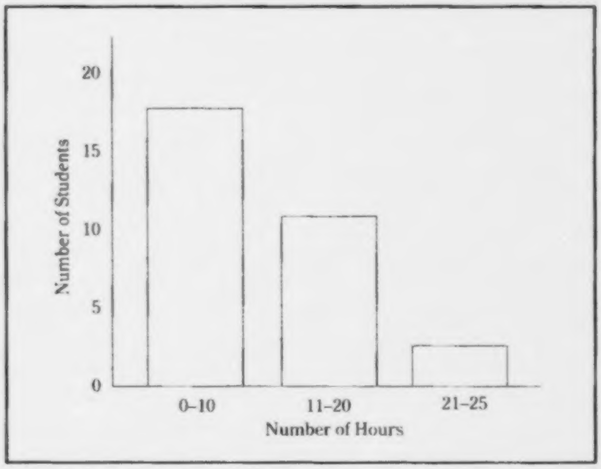


Fig. 34.2. Group 1's graph

I chose not to intervene in their discussion other than to ask clarifying questions. At this point, I hoped that the issue of inconsistent intervals would arise during whole-class discussions. I judged that it would be much more productive for all the students to be engaged in a discussion about this issue than for me simply to "correct" their mistake. My telling the students would not constitute a basis for their understanding.

As the groups finished their investigations, I asked each group to come to the chalkboard and present the results of their work. The first group to report was Latisha, Nathan, and Amee. When they finished, other students in the class appeared to accept their graph as a reasonable way to proceed and a very sufficient way to represent the data. Interestingly, in creating the graph, they modified their intervals so that they now ranged from 0 to 10, 11 to 20, and 21 to 25. This change occurred as they were placing the data into the three categories. When I asked where they placed 10.5, they explained that they had rounded each data point to the nearest whole number.

At this juncture, I did not point out the problem with rounding the data and the effect that it has on the representation. Further, I did not see it as my role to ensure that the students made the graphs "correctly" or used the representation that I had envisioned. Instead, I was much more interested in ascertaining what the students did know and thinking about how to use their current understandings as building blocks for later lessons. If these issues did not emerge as problematic for the students, I would be forced to impose conventions, such as how to "correctly" make the graph. Had I simply corrected what I perceived as a mistake, subsequent activities might have caused the students to focus on guessing what I wanted instead of allowing them the opportunity to reason independently. Further, I was still unclear whether the students viewed the data along a continuum of values or had simply categorized the data points. My imposing a continuous scale, as was necessary for the histogram, would be very problematic if the students, at this point, were not reasoning in that manner. For me, a better alternative was to use my knowledge of their current understandings to try to structure subsequent tasks that would make these issues the focus of the investigation.

The second group to share its results drew what I would classify as a modified histogram (see fig. 34.3). The members grouped their data into intervals with a range of five, placing the bar at the upper limit of the range of the interval. In addition, instead of making clearly defined bars, they used line segments. Further, in constructing their graph, they had used a large dot to mark the endpoint of the segment. Therefore, the length of the segment represented the number of data points that fell in the interval (i.e., from 0 to 5, from 6 to 10, and so on).

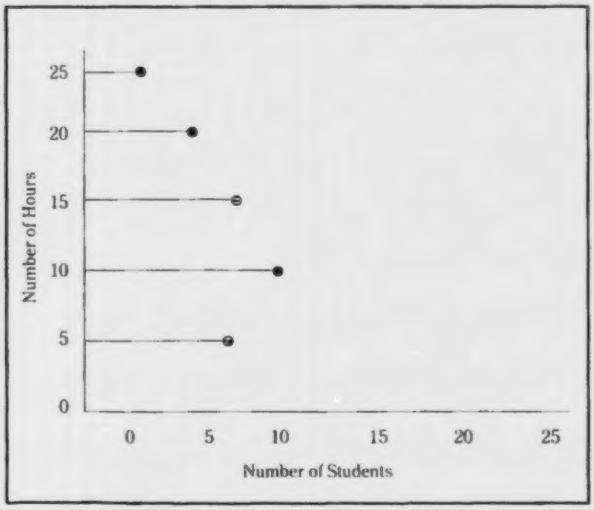


Fig. 34.3. Group 2's graph

The dot became an issue for other students in the class when I asked, "How are these two graphs alike, and how are they different?"

Andy: That's [points to group 2's graph] a line graph and that's [points to group 1's graph] a bar graph.

Teacher: That's a line graph, and this is a bar graph. Anything else?

Carla: I thought it was supposed to be bars and not like little lines . . . like bars.

Teacher: Why do you think it is bars?

Carla: Because if you do a line, it is supposed to go up at the time when it goes up, and it goes down when . . . it goes up when the rate is high and goes low . . . .

Lynn: I think it is supposed to have bars when it goes vertical [sic] like that.

Teacher: What if she is calling these skinny bars? What if she is saying that these are really just skinny bars?

Maggie: With dots on the ends of them?

Jose: With dots on them? I mean, you could do that, but you wouldn't have a line on them, you would just have the dots.

Paul: A line graph is supposed to be connected to other line segments.

For the students, certain conventions were associated with making graphs, and if you used "dots," then you must be making a line graph of connected dots. The rules for the use of dots in making graphs seemed very clear to the students; however, they appeared to take great liberties with what I had interpreted as histograms. Their notions of "school mathematics" became intertwined with their goal of making a representation of the data.

The third group to present their graph also drew what I would call a version of a histogram. They appeared very clear on the rule that the intervals all had to be of the same size. However, in their "histogram" the intervals were each composed of six of the thirty data points. They said that they decided to use six data points in each interval because you could divide 6 evenly into 30 to get five groups. They had ranked the data from least number of hours of television watched to the greatest, as if finding the median, and then divided the ranked data into five groups of six numbers each (see fig. 34.4a). They then totaled the six data points in each group, which gave them the height of the respective bar (see fig. 34.4b). The result obviously gave a series of bars with increasing heights, since the subsequent bars contained the larger data points (see fig. 34.4c). The group members labeled the horizontal axis according to which six of the thirty data points were contained in the interval, and they used the vertical axis as a scale for the total hours for each of the six groups of data points.

After group 3 finished explaining its graph, I again asked how the three graphs were alike and different. The students' discussions tended to focus on the direction of the bars (i.e., vertical or horizontal) and the labeling of the axes. For instance, many students noted that group 2 and group 3 both labeled the horizontal axis as "number of students." The fact that the number represented very different reports was of no consequence. They also noted that group 2 was the only group whose graph was "sideways." Their focus was on the superficial features of the graphs instead of the underlying meaning and intent.

The members of the next group to report stated that they had calculated the mean. I then asked why they thought that the mean was a good way to represent the data.

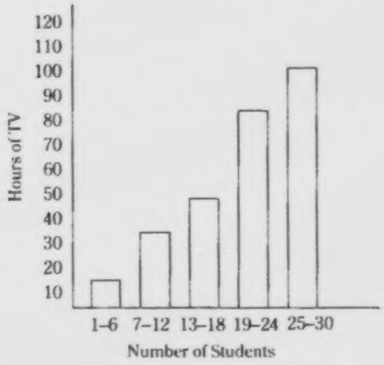
Mark: Well, we averaged it out, and it worked pretty good; well, because that's like saying, well, when you make a 100 on something and a 60 on something and they average that out, like on your report card, that's really not right because you made a high grade on one thing and a low grade on another.

0 1.5 2.5 3 3.5 4 / 4.5 5 6 6 8 8.5
/ 9 9 9 10 10.5 11.5 / 12.5 13 13.5 14 15 16
/ 16.5 18 19 19.5 21 23

(a)
Ranked data separated into five groups

0	1.5	2.5	3	3.5	4	= 14.5
4.5	5	6	6	8	8.5	= 38
9	9	9	10	10.5	11.5	= 59
12.5	13	13.5	14	15	16	= 84
16.5	18	19	19.5	21	23	= 117

(b)
Totals for the five groups of data



(c)
Group 3's "histogram"

Fig. 34.4 Group 3's data points, rankings, and graph

It appeared that in Mark's justification for the use of the mean, he was questioning whether it was "right," since as he related it to finding an average grade, he would not want to use the mean if he had a high score and a low score. I found it intriguing that as he offered his justification to the class, he appeared to be reconceptualizing his understanding of the mean as a way to represent his group's data. Of the remaining three groups, two calculated the mean and the third made a bar graph of each value in the data set.

As the class drew to a close, I realized that I had generated more questions for myself than answers. The students had obviously acquired ways to reason about the data. However, their ability to make judgments tended to be impaired by their notions about "rules," for example, if you use dots, they have to be connected. Also, group 3 was certain that each bar had to represent the same number of data points, but the intent of the graph was never clarified. Initially, I had been concerned not about whether the students could make the graphs correctly but with how they reasoned about the data set. However, as a result of interacting with the students on multiple tasks, I realized that their previous experiences with learning how to make graphs would clearly have a large impact on future instructional goals. Apparently, the set of rules associated with the construction of graphs formed the basis of their initial investigations with graphs. They perceived that I was now asking them to remember all the rules as they made reasonable judgments about the data sets rather than to reason and construct their own ideas. Asking them to create their own representation appeared to be a shift in the type of activity that they normally associated with sets of data. Also, their interpretation of the mean as a representative value was unclear to me. Students clearly knew how to calculate the mean, but their ability to talk about it as a reasonable way to represent a data set seemed limited.

Implications for Instructional Tasks

The results of my observations of the students' ways of reasoning on this and other tasks had serious implications for instructional "next steps." To support students' development of ways to reason logically about data, instructional tasks would need to build from the students' current understandings. Simply restating rules and procedures for the proper ways to construct types of graphs would not equip the students with powerful tools for data exploration. Their subsequent activity would be reduced to deciding which graph to use for which set of data according to some predetermined criteria. Further, problematic areas in their current understandings would need to be highlighted in the context of investigations so that those areas could then become the focus of discussion. This outcome would require finding situations in which the data presented the opportunity for students to reason about the implications of their decisions.

In working with my colleagues to develop a sequence of specific tasks that would capitalize on the students' current understandings, we decided to focus on the use of the mean as a representative value. We thought that this understanding could be best achieved by having students work to compare two sets of data to make a decision. In this situation, each choice would result in a consequence. We hoped that students would then come to see the value and limitations of the mean. As we deliberated, we decided to find situations in which the two data sets had very similar means even though the individual data points in one of the sets varied greatly. In these particular situations, it might be important to know more than just the mean; one might also need to know the consistency, or the range, of each set. Students would then need to work to find another way to "tell the whole story" of the data.

One such investigation involved comparing the hours of use of a sample of ten separate batteries from each of two different brands of batteries. Students were immediately engaged, since most of them depended on batteries for some type of personal electronic apparatus. As they began to analyze the two sets of data, their investigation led to discussions about which brand had "more bad batteries" and which brand was "more dependable." During the discussion, some students argued that if you used the battery with the higher average, over time that brand would be the better choice. The students then worked to find other ways to represent the information to support their arguments. This type of discussion provided opportunities to build from students' investigations to support conceptual understanding of the mean in relation to a set of data and of the mean in relation to a graph. The development of this type of task clearly requires that we find situations that not only appeal to the students but also feature data that fit with our pedagogical agenda. This approach is far more work-intensive than teaching from a textbook. However, I believe it to be both necessary and appropriate.

Conclusion

As a result of my interactions with these students, I have further refined my ideas about how to support students' development of statistical reasoning. I believe that students need to develop ways to reason logically about data—not memorize rules and procedures. For me, this belief involves students in tasks that permit the exploration of multiple forms of representation. Students' experiences with data, graphs, and representative values need to reflect the expectation that they reason about situations rather than simply apply rules and algorithms. For instance, students should use their analysis as the basis of arguments. By making reasoning the focus of our instruction, we then change our expectations. We focus on the meanings that graphs and representative values have for the students in light of their investigations.

In the past, the distinction I often made between "graphing" and "finding the average" caused students to infer that these two types of representations are not related. They therefore lacked the understanding that both representations are models of the data set, each highlighting different aspects of the set. By allowing students' initial exposure to data analysis to be in the context of exploration of data sets, we give purpose to the investigations and subsequent representations. The specifics of constructing graphs can emerge from the importance of clarifying the representation. For instance, the graph presented by group 1 offered an opportunity to build on students' contributions while clarifying the purpose of a histogram. In this setting, the problem of rounding data points could be highlighted as students wrestled with the notion of a continuous scale.

As a result of thinking differently about how graphical representations should be incorporated in the middle school curriculum, I see the value in allowing students to engage in analyzing data before they have mastered all the conventional "tools." In this way, the tools emerge from students' investigations, thereby acquiring meaning in the context of their activity.

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ACTIVITIES

Data Analysis and Probability

McClain: "Reflecting on Students' Understanding of Data"

In the article, the author raises the following pedagogical issues:

a) Allow students to engage in data analysis before they have mastered all the conventional "tools."
b) Do not correct students' misuse of graphing conventions as they make them, but design subsequent tasks that make these misuses the focus of investigation.
c) Make reasoning rather than the mastery of graphing conventions the focus of instruction.

1. Write a reaction paper to this article that addresses one of the issues above. State whether you agree or disagree with the author's point of view, and give reasons. Include information from the article to substantiate your position.

2. Describe how this article does or does not promote the goals listed for the Data and Probability Standard.

Section 3: Data and Probability Standard

Discuss how the three articles in this section do or do not illustrate growth across grades in the Standard's goal that all students should "formulate questions [and] collect, organize, and display relevant data to answer them." Address the different expectations for students at the different grades in the three articles.

Part 3:

Process Standards

ALTHOUGH the processes of learning and doing mathematics are described by a set of five separate standards, these processes overlap and reinforce each other. This interrelatedness is evident in the examples of classroom mathematics described in the following three sections: (1)Problem Solving and Representation, (2)Communication; Reasoning and Proof; and (3)Connections.

The articles selected for these sections provide a stimulus for reflecting on the separate goals listed for each Standard. However, these articles are not the only exemplars of each Standard. The goals listed here can also be identified in articles contained in other sections of this book. The entire collection reflects the pervasiveness of the Process Standards throughout all aspects of the mathematics classroom experience.

1.

PROBLEM SOLVING; REPRESENTATION

THE PROCESSES of problem solving and representation are intertwined. As the collection of five articles illustrates, the ability to represent mathematical entities and relationships in some form helps students make sense of and solve complex problems, as well as promotes deeper understanding of mathematical ideas.

Problem Solving

INSTRUCTIONAL programs from prekindergarten through grade 12 should enable all students to—

- *build new mathematical knowledge through problem solving;*
- *solve problems that arise in mathematics and in other contexts;*
- *apply and adapt a variety of appropriate strategies to solve problems;*
- *monitor and reflect on the process of mathematical problem solving.*

The processes of problem solving should permeate the classroom experience. Rather than consist of a set of isolated heuristics that are studied and applied only during "problem solving" activities, these processes are an integral part of what it means to learn and do mathematics. The three articles in this section present innovative uses of problem solving to both facilitate learning and understand students' thinking.

The article by Sawada illustrates how problem solving can be used, not as an end in itself but as a means for building new mathematical knowledge. The author describes a sequence of situations posed by a typical grade-5 Japanese teacher to introduce and develop the notion of density. The students began within a familiar context, finding a way to determine which of two rooms was more crowded, and moved from there to investigating the more complex problem of expressing the relative "crowdedness" of the islands Hokkaido and Okinawa.

Rachlin describes three basic processes involved in developing mathematical power. Reversibility, flexibility, and the ability to generalize are important skills that facilitate a student's ability to look for, understand, and learn the mathematics embedded in both routine and nonroutine problems presented in school mathematics.

The article by Kelly discusses the power of visualization to facilitate problem solving. In-service teachers developed their own abilities to use drawings and then created a sequence of activities to promote their own students' skills in this area. Examples of third-, fourth-, and fifth-grade students' work illustrate how children use drawings to promote and organize their thinking, as well as to communicate their solution strategies to others.

Mathematics as Problem Solving: A Japanese Way

by Daiyo Sawada

WE KNOW from many international studies that Japanese students, and Asian students in general, do quite well in mathematics on knowledge-level questions and on deeper conceptual and problem-solving items (e.g., Stevenson and Stigler [1992]; Stedman [1994]). A question arises: How are mathematical concepts and problem solving approached in Japanese elementary schools?

My response to this question was largely formed during a research visit to the University of Michigan in Ann Arbor at the invitation of Harold W. Stevenson during spring and summer 1993. His project team was in the midst of analyzing voluminous data gathered from classrooms in Taipei, Beijing, Sendai, and Chicago. Stevenson kindly made it possible for me to immerse myself in a set of classroom observations of 160 lessons taught by forty teachers of grades 1 and 5 in Sendai, Japan.

To share with you what I found, I do not want to provide a "once over lightly" treatment of a few general characteristics of problem solving in Japanese elementary schools. Rather, I have selected a typical grade 5 lesson taught by a typical grade 5 teacher with typically forty students in her classroom. This teacher does not have many striking characteristics that would mark her as exceptional among the set of forty teachers included in the data set. I would describe her as competent, careful, and caring. That description covers most of the other teachers, as well. I present the lesson in some detail to give readers a feel for the problem. I then identify and comment on some aspects that were particularly striking to me. I close the article with a question and a partial answer to the main question. [Note: Parenthetical material is enclosed in square brackets.]

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Lesson Description (Grade 5)

The teacher, Ms. Sato (a pseudonym), introduces the lesson with a discussion of "crowdedness," a notion quite meaningful for people of Japan. After pointing out that one cannot say that a place is crowded simply because it has a lot of people, she focuses the discussion on a particular comparison: "Which is more crowded, Hokkaido or Okinawa?"

She reminds the class of problems that they worked on in previous classes and raises one of them again: "Which tatami room is more crowded, the one that measures 10 mats and has 1 person or the room that measures 10 mats and has 10 persons?" [Note: The area of a tatami room is measured in terms of the number of tatami mats required to cover the floor. The floor space of a tatami room is always designed as a particular tessellation of tatami mats. A tatami mat itself is 1 m by 2 m.] A student volunteers a response: "The second room." Ms. Sato agrees and explains that this problem is easy because the rooms are the same size: "But in our case Hokkaido and Okinawa have different areas." She then asks if the two areas can be compared. A student suggests figuring out how many times bigger Hokkaido is than Okinawa. The teacher explains that this method is similar to the one used in the "quantity of salt experiment" they worked earlier and that this approach too can be a good method.

Another student suggests that one can "turn over a certain amount of area from Hokkaido to Okinawa to make them the same size." Other students object, suggesting that this method is too cumbersome to use here. Another student volunteers, "Like what we did with the tatami-room problem, we should try to figure out crowdedness for the same amount of area first," instead of messing around with exchanging area, as in the previous suggestion. Ms. Sato takes up this suggestion, asking, "What then should be done with the tatami-room case?" A student responds, "Compare number of persons per mat." The teacher writes this suggestion on the chalkboard while repeating it orally.

Ms. Sato recalls the figures from the tatami-mat problem from a previous lesson: room A—nine persons in a six-mat room, and room B—fifteen persons in a fifteen-mat room. She calculates the number of persons per mat for each tatami room and then asks, "Which room is more crowded?" She confirms the answer by drawing two diagrams on the chalkboard, showing one and one-half bodies per mat for room A and 1 body per mat for room B. She summarizes by saying that "we calculated quantity per unit."

Turning to the Hokkaido-Okinawa problem, she asks what unit to use. A student responds, "One square kilometer." Accepting this answer, the teacher points out again that they can base the calculation on the number of persons per square kilometer. She then summarizes with, "Population divided by area equals population per square kilometer" and writes this statement on the chalkboard. Students copy it into their notebooks. The teacher explains further that by dividing population by area, one figures out how many people live in the same amount of space.

In preparation for the calculations, the teacher reminds students about the number of significant digits to keep when rounding. A student responds, "Round off to significant digits when doing calculations" and explains that the numbers given in the problem are already rounded off to two significant digits, and the teacher confirms. She tells students to solve the problem by starting with Hokkaido, and she helps by having students identify the rounded numbers that they should use and writes them on the chalkboard. Students work on the calculations at their desks. Ms. Sato circulates. She then asks for an answer and gets "70.8" from a student. She tells him to round it off and then states that Hokkaido has about 71 people per square kilometer.

Students then calculate by using the numbers for Okinawa, and the teacher again reminds them to use two significant digits when calculating and not to round off until after they find the answer. A student volunteers "478," and Ms. Sato asks, "Rounding this number to two significant digits, we have what?" A student responds with "480." The teacher then asks which island is more crowded, receives the answer that Okinawa is more crowded, and confirms this response by reiterating that Hokkaido has 71 people per square kilometer, whereas Okinawa has 480 per square kilometer.

Ms. Sato directs students to look at page 79 in the textbook, and asks, "What do we call this 'crowdedness' that we are trying to figure out?" In a choral manner students respond, "Population density." The teacher confirms, saying, "We have just learned to figure out population density today."

At this point, thirty-five minutes have elapsed. During the remaining seven minutes, students do two more problems involving population density. [Two days later, students were still working on density problems but with some variation: the opening problem involved iron and silver—250 cc of iron weighing 1975 g; 350 cc of silver weighing 3675 g. This problem took twenty-five minutes to solve and discuss. The next day, the concept of average was developed as a particular case of density. The problem consisted of comparing the output per factory of two kinds of production where the procedures for handling density generated a quantity that could be called the "average production" (per factory).]

Looking Back

On reflection, these points seem worthy of notice:

- A deliberate effort was made to connect the present problem to previous problems and solutions.
- Simpler problems were used.
- The teacher functioned as a guide with a definite agenda.
- Crowdedness provided a meaningful context for embedding the concept of population density.
- In turn, the concept of density provided a context for embedding the concept of average.
- More than thirty-five minutes was spent working on one problem.
- Student contributions were used to determine the content of the lesson as well as its flow.

- The notion of rounding and significant digits was used consistently when doing the computations.
- The problem was rich in social studies content.
- Formulaic statements did not enter the lesson until they could be used as summary confirmations.
- The teacher did not follow up the lesson by assigning several application exercises. Instead, two related problems are discussed and solved, each widening the interpretation of the idea of density.

Commentary

The comments to follow also reflect my study of the 159 other lessons taught by Sendai teachers in the data set.

Embedding

Problem solving pervades this lesson. Problem-solving techniques are used not only in solving the problem about density but also in the approach to the lesson structure itself. I have used the term *embedded* to describe this more pervasive use of problem solving in teaching mathematics in which problem solving is the medium for mathematics teaching and learning. In the lesson described, we see as much emphasis on the medium—problem-solving process—as on the message—the particular mathematical content to be learned. This emphasis on the medium in many grade 5 lessons in Sendai often takes precedence over the message so that instructional decisions—how to respond to mathematical "errors," for example—are made with the health of the medium as much in mind as the need to assess the correctness of the message. For example, a so-called "error" is often discussed at length and thereby contributes to the diversity of the solution space and therefore to the health of the medium, and thus it is valued even if not particularly "correct" as a message. Indeed, such contributions, because they do enrich the solution space, are actively sought by teachers through the repeated use of such simple leads as "Any other methods?" None of the responses to such questions is rejected at the outset, and this decision not to reject indicates that the health of the medium is at least as important as the correctness of the message. This priority on the medium occurred in the lesson in many particular as well as general ways.

From a mathematical viewpoint,

- crowdedness was used as a medium for (to embed) density;
- density was used as a medium for rate/ratio; and
- in its turn, rate/ratio was used as a medium for average.

From a pedagogical viewpoint,

- the general problem situation was a medium for the structure of the lesson;
- the particular problem was a medium for the mathematical concept; and
- the multiple solutions generated by students were a medium for the "correct" solution.

From a learning viewpoint,

- the process of solving the problem was a medium for learning mathematics;
- right or wrong, student answers were the media for exploring the solution space; and
- generating and discussing solutions were the media for arriving at and critiquing answers.

Spending thirty-five minutes or more on one problem

Spending nearly the whole period on one problem might seem an inefficient and drawn-out way to teach a mathematical concept. However, if only four or five minutes were spent on a nonroutine problem, I would have grave doubts that a problem-solving approach was being used at all. For a rich problem-solving process to occur, time must be allowed for investigating the nature of the problem, generating and proposing multiple solutions, discussing and critiquing various types of solutions, assessing and comparing the relative merits of each solution within the set of solutions, and reflecting on what one has learned and what else could be learned. This method sounds like a lot of rhetoric, but it happens regularly in most grade 5 classes in Sendai, and it means that the whole period will often be spent on only one problem. Statistically speaking, 86 percent of Sendai grade 5 teachers use problem solving to embed mathematical concepts; the comparable figure for Chicago elementary schools is 14 percent.

Acting out the problem

Although the students in the lesson described did not act out the solution process, in other Sendai classrooms this activity happened quite often. For example, the teacher would bring cushions to simulate tatami mats, children would select which one(s) they would prefer to sit on, and a problem about density would be enacted. Rooms with so many mats would next be drawn on the chalkboard, with students selecting which room they preferred. The density problem thus enacted would be ready to be solved.

Multiple solutions

Multiple solutions are the "rice and fish" of the problem-solving approach used in Sendai. If only one solution exists, it would have to be the correct one. In this situation, nothing would function as, or need to function as, a medium for anything else. The correct solution would be demonstrated or illustrated for all to learn. Multiple solutions—and establishing the conditions for generating, articulating, understanding, comparing, and critiquing multiple solutions—are required components to the way problem solving happens in Sendai classrooms.

Interesting problems to solve

Although the problem of the relative population density of Hokkaido and Okinawa may be of interest to Japanese students, as a generalization, most problems are rather mundane. Many of them remind me of typical word problems found by the hundreds in Canadian textbooks. For the classrooms in Sendai, the problems per se are not different; what is different is the way that these problems become the medium of instruction and of learning.

Using manipulative aids

The NCTM publication *Making the Grade in Mathematics: Elementary School Mathematics in the United States, Taiwan, and Japan* (Stevenson et al. 1990) reports a surprising finding: When manipulatives are introduced into United States lessons, the amount of talk decreases, whereas in Japanese elementary classrooms, it increases. This finding may have been surprising, but when problem solving is the pervasive mode of teaching and learning, introducing manipulative aids contributes immensely to the variety of multiple solutions to be generated, assessed, compared, and so on, thus providing many more reasons and opportunities for talk.

Interpretation rather than application

Currently in North America, we talk about developing problem-solving strategies and skills and then applying them. In this sense, problem solving is split into two rather distinct parts:

- The learning of the concept and skills, perhaps through problem solving
- The use or application of these concepts and skills in similar situationS

This second part is often taken to be the full extent of problem solving. It is a matter of applying concepts learned, not of concepts and skills. Our textbooks are organized this way; concepts are taught—often by demonstration or explanation, as well as by problem solving—and then students are given a collection of similar "problems" to do. Because of this practice, lessons in North America are likely to end with students working at their desks. The contrast in Sendai is striking. Classes often end with discussion. And when children are working at their desks at the end of the class, they are not only applying the concepts just learned but also interpreting problem situations that extend the ideas beyond the initial circumstances. In the lesson presented, the problem-solving approach is not two parts but just one. On the one hand, problem solving becomes two parts when the concepts learned—the messages—become so important that they need to be separated and dealt with differently—as applications. On the other hand, if we keep the problem-solving process intact and pervasive, the messages learned will never dominate the medium that created them.

Conclusion

Several other aspects about problem solving in Sendai deserve mention, such as the teacher's deliberate making of errors or the occurrence of memorization, but I think that the important points have been made. I close this article with a question and a partial answer. Why has the problem-solving approach as a medium taken hold so pervasively in Japanese elementary schools but remains largely problematic in North American schools despite a decade in which nothing in mathematics education received more attention than problem solving? Although not wishing to oversimplify, I suggest that in Japan, accepting ways of doing things as being as important as the things themselves is a familiar and comfortable stance. We see it in martial arts, flower arranging, the tea ceremony, pottery making—and in teaching, too. Problem solving as a medium for teaching mathematics is another of these important ways.

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Learning to See the Wind

by Sid Rachlin

WHEN I was at the University of Hawaii, a colleague told me about an experience that he had had as a first grader growing up in Hawaii nearly forty years earlier. It was a simpler time, one in which a trip from one side of the island to the other was no small feat, since it involved traveling over a narrow mountain road called the Pali that stretched through a pass in the mountain range. At the top of the mountain, his family stopped to enjoy the view. As he watched from the windy edge of the mountain's lookout, he made a discovery that he deemed worthy of sharing at show-and-tell the next Monday. Contrary to what he had been taught in class, he *could* see the wind!

On Monday, he eagerly awaited his turn to share his discovery. But to his disappointment, when he told the class of his adventure, rather than share in his excitement the teacher corrected him. "Now David," she said, "you know that you cannot see the wind. What you saw was the dust and dirt being blown around by the wind—the wind is invisible."

Two things struck me about my colleague's tale. First, how meaningful this experience must have been for him to have held it in memory for nearly forty years. What made this incident such a *memorable* experience in his life? And second, his story made me think about how mathematics is like the wind.

As my colleague spoke, my mind wandered to thoughts of the old Claude Raines movies of the invisible man. How did one catch an invisible man? In one Abbott and Costello version, the invisible man was captured in a kitchen when someone tossed flour in the air. Amid the flour, the invisible man became visible and was caught. But what exactly was seen? After all, the invisible man was by definition invisible, so all we saw was flour.

And what of mathematics? Are the symbols of mathematics really mathematics, or are they merely the bits of flour that we throw about to help us find mathematics? Too often mathematics is taught as if the flour is what is important. Yet all the symbols—even the manipulatives, calculators, and computer software—are only flour. The pages of dittoed ink do not make mathematics. The base-ten sets and counting aids do not make mathematics. The textbooks and technology, the manipulatives and worksheets provide opportunities for our students to see mathematics—to see the invisible man. But only if our students are looking for mathematics.

Mathematical power comes from the development of students' beliefs that mathematical understanding is something for which they are responsible and is something that they can expect to achieve. The learning of mathematics, beyond the level of rote memorization of formulas and algorithms, can be regarded as a kind of problem-solving process. That is, even the application of formulas to *routine* textbook exercises involves some degree of problem-solving activity on the part of most students, at least for a while. Accordingly, the National Council of Teachers of Mathematics (1989) proposed that problem solving not be treated as a distinct topic but rather be viewed as a process that should permeate all mathematics instruction by providing "the context in which concepts and skills can be learned." The successful study of mathematics requires a set of problem-solving processes that help students reason mathe-

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matically, that offer a context in which students can communicate their reasoning, and that provide opportunities for students to link their learning to prior mathematical contexts.

Three of the basic processes identified as starting points are included in Krutetskii's (1976) model of mathematical abilities—reversibility, flexibility, and an ability to generalize. A first step in having students learn to see the wind is to restate routine textbook exercises as problems that can be used to help students reflect on their learning and develop their mathematical abilities, as in figure 36.1.

1. What number divided by 24 equals 3/4?
2. What number multiplied by 2/3 equals 3/2?
3. Hot dogs cost $1.25 and large soft drinks cost $0.75 at the carnival. Jerry has to buy hot dogs and large drinks for eight people. How much money does he need?
4. Find three numbers less than 100 whose product has four 0's at the end.
5. A pair of sneakers was originally priced the same at three different stores in the mall. As luck would have it, all three stores put the sneakers on one-day-only sales for the day after Thanksgiving. The first store has the price marked down 30 percent, but from 7 A.M. to 10 A.M., you will get an additional 25 percent off. The second store has the price marked down 25 percent, but from 8 A.M. to 11 A.M., you will get an additional 30 percent off. The third store has the price marked down 50 percent from 7 A.M. to 11 A.M. Which deal is best for you?

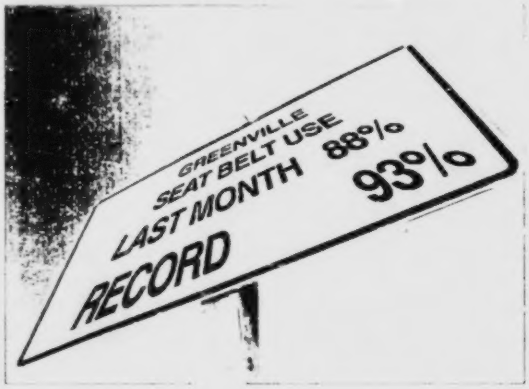
Fig. 36.1. Analyzing various problems for content

Reversibility is an ability to restructure the direction of a mental process from a direct to a reverse train of thought. For example, in the expression □ + △ = ○, we might be given values for □ and △ and be asked to find a value for ○. The reversibility of this addition incorporates three variations: where the values of □ and ○ are given and the value of △ is to be found, where the values of △ and ○ are given and the value of □ is to be found, and where the value of ○ is known and both □ and △ are to be found.

To create a nonroutine *routine* problem, we need only to switch the given information of a standard textbook problem. For example, "Simplify 4^4" becomes the following:

> What number raised to the fourth power equals 256?
> Four raised to what power equals 64?
> Find three integers that raised to a power greater than 1 equal 256.

Although the new problems are not much different from the original, the richness of the class discussions is quite different. Much of precollege mathematics refers to actions that can be reversed, including, but not limited to, the actions of graphing, making a table, solving an equation or problem, and simplifying an expression by performing indicated operations. Rather than refer merely to working backward through the steps of an algorithm, reversibility in thinking refers to an ability to find an alternative path for meeting the conditions of the problem. To create such a problem may be as simple as finding the mathematical answers presented in the world around us and asking students how the answers might have been obtained (see fig. 36.2).



As a part of the Click-It or Ticket campaign, the signs throughout Greenville report seat-belt use. If we assume that numbers have been rounded to the nearest percent, what is the smallest number of cars that could have been sampled to get the record seat-belt use of 93 percent? If the same number of cars are sampled each month, what is the smallest number of cars that might be sampled to get both 88 percent last month and a record of 93 percent?

Fig. 36.2. Click-It or Ticket

Flexibility is the ability to switch from thinking of one method of solving a problem to another. How a student perceives a problem shapes the other processes that she or he may bring to bear on the solution of the problem. The various solution paths that a student selects establish the structure for the problem. For example, problem 1 in figure 36.1 has a wide variety of appropriate solution paths depending on the way in which the task is perceived, for example, as equivalent fractions, a proportion, a division problem, an equation, and so on. Students who have only one way to see a problem are limited in their ability to solve it and to link it to other mathematics.

In solving problem 2 (fig. 36.1), many students simply multiply 2/3 × 3/2, whereas others are unsure whether to represent the problem as 3/2 ÷ 2/3 or 2/3 ÷ 3/2. Students tend to be rule oriented and state generalizations about the solution process before beginning the problem. In many situations, students falsely generalize how to do the problem, overgeneralizing either about something that occurs in the problem or about something they have heard their teacher say. Students have solved the problem correctly in several different ways. (See fig. 36.3.)

Students changed the way they solved the problem when the problem

$$\text{“Multiply } \frac{2}{3} \times \frac{9}{4}\text{”}$$

was changed to

$$\text{“What number multiplied by } \frac{2}{3} \text{ equals } \frac{3}{2}\text{?”}$$

Each student's method is rich with opportunities for discussion. Students' errors provide opportunities to pursue new directions or to clarify misconceptions (Rachlin 1998).

The way in which a student solves a problem dictates whether the problem is helpful in the development of the student's flexibility. The fact that many ways exist to solve a problem does not mean that an individual will learn to search for other solutions unless the problems presented guide the learner explicitly or implicitly in that direction. Problem 3 in figure 36.1 uses the students' natural solution approaches to lead into a class discussion of the distributive property. Only through discussion can students come to realize that another student's way to solve the problems is viable. Likewise, a teacher may need to guide the connection between the students' solutions and the mathematical importance of the generalization they are making.

Kathy solved the problem like an equation. First she wrote

$$\frac{2}{3}n = \frac{3}{2};$$

then she multiplied both sides by 3/2.

Curtis found common denominators for the 2/3 and 3/2 and then rewrote the problem as

$$\frac{4}{6} \bullet (\ ) = \frac{9}{6}.$$

But 6 • 1 = 6 and 4 • (9/4) = 9, so the answer was (9/4)/1.

Margaret multiplied the 2/3 by 3/2 to get 1. Then she multiplied the 1 by 3/2 to get 3/2. She wrote this process as

$$\left(\frac{2}{3} \bullet \frac{3}{2}\right) \bullet \frac{3}{2} = \frac{3}{2}.$$

The fraction she was looking for was

$$\left(\frac{3}{2} \bullet \frac{2}{3}\right),$$

or 9/4.

Lani wrote the 3/2 as an equivalent fraction such that 2 would divide its numerator and 3 would divide its denominator. She chose 18/12 as a fraction that met the conditions. Then

$$\frac{2}{3} \bullet ? = \frac{18}{12}.$$

Since 18 ÷ 2 = 9 and 12 ÷ 3 = 4, she decided that her solution was 9/4.

Joe wrote

$$\frac{2}{3} \bullet (\ ) = \frac{3}{2}$$

and then filled in the parentheses. Since he wanted to get a 2 in the denominator, he needed in the denominator of the fraction in parentheses a 2 that would "cancel" the 2 in the numerator of the 2/3, so he needed a 4 in the denominator of the fraction in parentheses. He also said that he needed a 9 in the numerator. Why?

Fig. 36.3. Students use various strategies to solve the same problem.

The ability to generalize includes both the ability to deduce the general from the particular cases to form a concept or make a generalization and the ability to subsume a particular case under a known concept or generalization. One technique for having students form their own generalizations is to have one student predict another's answer. Problems of the form $n + s = I$, where only general conditions are given, lead nicely into "what if" explorations. For example, in the problem "Find two fractions whose product is an integer," what if the product is a proper fraction? What if the two fractions are improper fractions?

Besides making generalizations, the ability to generalize refers to a student's ability to recognize that a particular situation fits the conditions of a known generalization. Problem 4 in figure 36.1 can be used to extend some of the generalizations that students may have made in their introduction to number theory. Problem 5 links their work with percent to the power of the associative property. For students to develop an ability to generalize, it is not enough that they see and apply generalizations; they must also learn to make and test their own generalizations.

Unlike the nonroutine problems and applications that must be searched out in calendars, conference presentations, professional publications, and mathematics texts, nonroutine routine problems can easily be created by the teacher. To develop students' mathematical abilities, begin by selecting a standard exercise from the textbook. Solve the task. Then pretend to "spill" something on the work. What results is a richer problem for students to solve.

Amazingly, this simple technique presents an opportunity for students to develop a deeper understanding of the mathematical procedures they encounter. If the routine task was to find the mean, median, and mode for a given set of scores, then after a "spill" the task becomes the nonroutine problem in figure 36.4.

Mark was surprised to find that the mean, median, and mode were the same for the five scores his teacher had given him. Unfortunately, before he could show his mom his work, he spilled hot chocolate on the paper. All that remained were three of the five scores: 85, 65, and 70. What pair of scores might be chocolate covered? How many pairs of scores will meet the conditions of the problem?

Fig. 36.4. A nonroutine *routine* problem

The challenge of teaching mathematics is to provide an environment in which students can see, touch, understand, and learn. The learning environment that we offer will set the probability of students' understanding, re-creating, applying, and retaining the patterns of mathematics. Although manipulatives, computer software, cooperative learning, and portfolios may be used by students to facilitate their learning, the role of instruction is to coordinate the alternatives and maximize learning. To this end, instruction and curriculum are inseparable. The curriculum must be organized to enhance the opportunities for students to learn. The instructor must present the curriculum in a manner that increases the learning potential. A first step to having students understand mathematics is to design problems that can be used to help them reflect on their learning and to create a classroom in which students learn to see mathematics.

Challenge yourself to do the following:

- Begin class tomorrow with a nonroutine *routine* problem on something you have taught this year or are planning to teach tomorrow.
- Listen to your students, and try to use their thinking as a tool for facilitating your instruction.
- Share your experiences with a colleague.

If you create a problem that works well at helping students understand mathematics or helps you understand your students' understanding, share it with others by submitting it to the "Readers Write" or "Teacher to Teacher" departments of *Mathematics Teaching in the Middle School.* By creating these kinds of problems for your students, you, too, can help them "see the wind."

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Improving Problem Solving through Drawings

by Janet A. Kelly

WHILE working with third-, fourth-, and fifth-grade teachers in a National Science Foundation–sponsored project designed to enhance the mathematics and science teaching of in-service elementary teachers, we recognized that teaching mathematics problem solving was one of their greatest challenges. Discussions with the teachers revealed that most were using an algorithmic approach to problem solving with an emphasis on facts, rules, and procedures. Their students were being taught to solve word problems in a systematic, single-mode manner. We found that the teachers were most comfortable with the algorithmic approach because that is how they were taught mathematics when they were in school. As one teacher commented, "I was stunned to find out that not everyone worked math problems the same way." Teachers admitted that the algorithmic approach posed obstacles for their students. Although many students could learn the procedure to solve a particular type of problem, frequently they could not solve related problems. The teachers sensed a lack of conceptual understanding and transfer on the part of the students. One teacher understood the uncertainty that her students faced in working word problems: "My problem in math is that I learn it only one way, and then I can't apply what I've learned to another situation. My [students] do this, too."

On the basis of our own learning and teaching experiences and published evidence that problem solving is facilitated when students are encouraged to visualize (Hembree 1992; Lester 1982; Wheatley 1991; Wheatley and Brown 1994; van Essen and Hamaker 1990; Yackel and Wheatley 1990; Yancey, Thompson, and Yancey 1989), we recommended that the teachers use visualization to help students solve mathematics. Noting differences in visualization techniques reminded teachers that more than one way to solve a mathematics problem is possible.

The Teachers and the Visualization Process

To introduce teachers to the power of visualization, we presented them with a problem and asked them to solve it.

> A snail is at the bottom of a 10-foot well. Every day he crawls up 3 feet, but at night slips down 2 feet. How many days will it take for him to get out of the well?

Without exception, all the teachers drew a picture to solve the problem, and although all the pictures were different, the teachers arrived at the same answer. Several teachers commented on the differences in the individual drawings, which prompted another teacher to say, "Just because we're all teachers doesn't mean we think alike. Besides, we all got the same answer." To encourage the teachers to draw pictures as a part of the problem-solving process, we presented many problems for which an algorithm was not obvious. After each problem set, the teachers shared and analyzed their drawings and solution strategies.

Gradually, the discussion shifted from the teachers' problem-solving ability to that of their students. The teachers agreed that visualization could be a powerful

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tool to enhance their students' problem-solving ability but questioned their ability to teach students to use this tool. Two teachers noted that they had tried to teach their students this method, but neither of them could decide on an effective teaching strategy. Others doubted their ability to teach students how to represent their mental images through the construction of a picture, and some were skeptical that elementary students could visualize mathematics problems at all. One teacher faced an additional obstacle: she had stressed neatness so much that those students who used drawings would erase their work before turning in their mathematics papers because they were fearful of being graded down for messiness. As one teacher said rather succinctly, "It's not enough just to tell my students to draw a picture." The teachers wanted specific how-tos.

Rather than offer suggestions that we had derived through our own research or had uncovered in the literature, we decided that the teachers should experience the value of visualization for themselves and then share their teaching tips with one another. For six weekly sessions, we followed a similar routine. We gave the teachers three or four "visualization friendly" problems. After the teachers solved the problems, we spent time discussing the various thought processes, reasoning, illustrations, and strategies that they used. These conversations were important to help the teachers become aware of their own thought processes, as well as the thought processes of their peers. The discussion gave the teachers insight into the mental processes and strategies that their students might encounter as they solved problems. This sharing emphasized the value of accepting a variety of solutions instead of focusing on only one solution strategy. We stressed that drawing pictures to represent visualization is only one way to promote the mathematical processes necessary to complete the problem.

The Students and What Works

In the beginning, the teachers had some difficulties and frustrations in implementing visualization in their classrooms. The most common problem was the inability of students to organize their thoughts well enough to visualize a problem and create an illustration. To address the issue, teachers agreed to ease students into visualization by supplying them with "picture" problems so that it was not necessary for them to draw their own. The teachers took the same problems that they had worked in their weekly meetings, created pictures for the problems, and asked their students to solve them. The teachers observed that the students readily used the pictures to solve the problems. One teacher reported that a student announced that it was "more fun to do math with pictures." The pictures gave the students something from which to work and served as examples for future visualization endeavors.

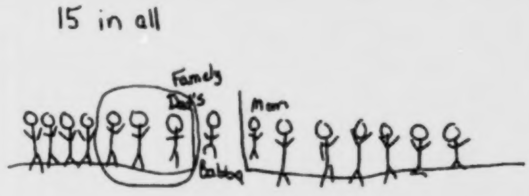
Following the success with the picture problems, the teachers asked their students to rewrite some mathematics problems and create their own visuals to further their understanding of what was being asked. In another session, students were asked to develop new problems and draw pictures that paralleled the problems they created. Then the students critiqued one another's problems. Several teachers reported that their students especially enjoyed this activity because they not only got to solve their classmates' problems but also got to identify any flaws.

Once the teachers believed that the students were comfortable with various forms of visualizations, such as diagrams, charts, and drawings, they presented their students with problems that contained no visual cues. Students were required to create their own visualizations from what they "saw" in each problem. Teachers reminded their students that they did not have to be artists to visualize and that stick figures or other simple configurations would perform the same functions as detailed, artistic renderings. For example, we gave students this problem:

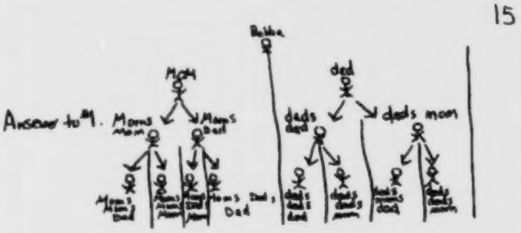
> On Baby Bubba's first birthday, he was given a party attended by his parents, his grandparents, and his great-grandparents. All of his grand parents and great-grandparents are still living and were able to come to the party. Including Baby Bubba, how many people were at the party?

Most students drew family trees or assigned names to each parent or grandparent to determine the correct answer. See figure 37.1. Three fourth-grade teachers reported that several of their students did not realize that each parent had his or her own set of parents; instead, they doubled up on one set of grandparents and arrived at 11 for an answer. One student personalized his problem and told the teacher that fourteen people were at the birthday party because one grandparent had died.

Teachers also presented the snail problem to their students. The types of pictorial representations that students used to solve the problem were more varied.


15 in all
Famely
Dads
Mom
Babba

(a)
This student placed Baby Bubba in the middle of Mom and Dad and added respective family members to each side.


15
Bubba
Mom
dad
Answer to #1.
Moms mom
Moms Dad
dads dad
dads mom

(b)
This student designed a flow chart to place family members.

Fig.37.1

Their visualizations ranged from number lines and charts to numerical operations (see figs. 37.2 and 37.3).

A student drawing illustrates the snail's daily movements to climb out of the well.

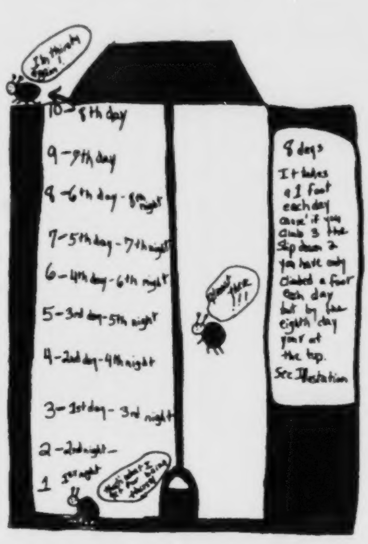

10 - 8th day
9 - 7th day
8 - 6th day - 8th night
7 - 5th day - 7th night
6 - 4th day - 6th night
5 - 3rd day - 5th night
4 - 2nd day - 4th night
3 - 1st day - 3rd night
2 - 2nd night
1 1st night
8 days
It takes a 1 foot each day cause if you climb 3 the slip down 2 you have only climbed a foot each day but by the eighth day your at the top.
See Illustration

Fig.37.2

The student created a "think sheet" with the larger numbers representing the number of days that the snail took to climb out of the well.

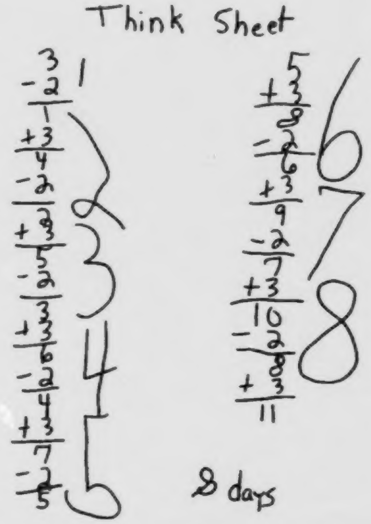

Think Sheet
8 days

Fig.37.3

After each problem-solving session, the teacher and the students discussed their solution strategies. The students were encouraged to share their drawings with classmates and to explain what they did and why. By the last two weeks of the project, teachers observed that it was no longer necessary to require students to make visualizations—they were doing it on their own. However, on one occasion, more than half of the 128 students involved in the project made no drawing or diagram to solve the following problem because they believed that it was "too easy":

> It takes three minutes to saw a log through. How long will it take to saw the log in four pieces?

One teacher decided that her students, even the ones who made drawings, "got hung up on multiplying 3 times 4 and just didn't think." Another teacher thought that it taught her students a valuable lesson: even if they think that they know the answer, it is better to check it out and make sure.

Each week, teachers brought their students' work to project meetings. The teachers reviewed and analyzed the drawings and discussed their observations and experiences. For the teachers, an unexpected benefit was that the drawings became a way to let the teachers "see" what was in the minds of the children.

Many teachers commented that the students enjoyed the variety of problems and the multiple solution strategies. The teachers expressed unanimous acceptance of using visualization as a strategy to enhance the problem-solving ability of their elementary students. All indicated that they will continue to use this method. That the teachers increased their enthusiasm for teaching mathematics—and the students, for solving problems—is also noteworthy.

Our observations suggest that young children's visualization is a natural process that can be nurtured by teachers without the need for special artistic skills or sophisticated techniques. We further note that students as young as 7 or 8 are capable of using visualization for problem solving. Visualization is a powerful tool for enhancing problem-solving ability. Its use may reduce mathematics anxiety in both teachers and students and, for the students, create a positive feeling about mathematics at an important developmental stage.

The basic tenets of visualization—that is, understanding what the problem asks and laying it out for solution—are applicable to any mathematical question. We urge teachers to explore this procedure as an aid in problem solving and teacher educators to incorporate the topic into their mathematics-education courses. We encourage all teachers to adapt the procedures described here to develop strategies for teaching visualization to their own students.

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Representation

INSTRUCTIONAL programs from prekindergarten through grade 12 should enable all students to —

- *create and use representations to organize, record, and communicate mathematical ideas;*
- *select, apply, and translate among mathematical representations to solve problems;*
- *use representations to model and interpret physical, social, and mathematical phenomena.*

As with problem solving, representation is an integral part of doing and learning mathematics. As the two articles illustrate, representation allows students to gain access to, explore, and communicate about complex mathematical ideas in ways that deepen and enrich their understandings. The students' experiences described by the authors also emphasize that the "ways in which mathematical ideas are represented are fundamental to how people can understand and use those ideas" (NCTM 2000, p. 67).

O'Brien and Moss posed complex problems to fifth- grade students and observed the range of solution strategies that were used. The authors found that those children who drew pictures were able to solve the problems successfully. The pictorial representations helped the students coordinate the separate pieces of information given in the problem and keep several things in mind at once as they worked toward a solution.

Bay, Bledsoe, and Reys describe an extended eighth-grade investigation of states in the United States that incorporated multiple forms of representation into an in-depth exploration of the notion of area. In this project, students compared the relative sizes of states by using tables of areas, scale models, and manipulations of the states' shapes using The Geometer's Sketchpad. These different representations encouraged students to use a range of skills, draw up and deepen related mathematical ideas, and engage in genuine problem solving.

On the Keeping of Several Things in Mind

**by Thomas C. O'Brien
and Ann C. Moss**

FOR THE past several years, an unusual teaching-learning situation has taken place in two fifth-grade classes: a university professor and the classroom teacher have been doing most of the learning. An initial opportunity for the professor to field-test some problems for a textbook that he was writing has turned into a long-term investigation of how youngsters generate and apply problem-solving strategies in various situations. Over the years, students have been exposed to nontraditional problems and asked to explore them individually, in groups, or together in one large group, depending on the situation. Wonderful observations have been made and used by both the professor and the classroom teacher to improve instruction.

The problems being shared here were posed to the students in midyear, after a semester of working with the professor. The students had had time to become comfortable with him and the kinds of problems that he would share for their investigation. Most of the students in both classes enjoyed these activities, but it must be noted that for some students, being asked to generate and apply their mathematical skills to a nontraditional problem was unsettling because their previous mathematical diet had been arithmetic. Nevertheless, many students in this group became better problem solvers with much more confidence in their abilities by the end of the school year.

The focus of this problem-solving activity, which took three sessions, was to find out how the students would approach a problem in which several conditions had to be met. The three lessons were conducted in the following sequence.

Lesson 1: Parakeets and Elephants

The teacher presented the students with this problem:

> The Clements's farm has parakeets and elephants. The Clements family includes twins, Tom and Ann. One day, Tom counted all the animals' heads. "There are 20 heads," he reported. Not to be outdone, Ann did her own counting. "There are 64 legs," she said. How many parakeets and how many elephants do the Clements have on their farm?

Students in both fifth-grade classes, taught in two consecutive afternoon class periods, eagerly went to work. In accordance with the classroom teacher's instructions, they worked in pairs for about forty minutes. No instructions were given except for restating and paraphrasing the problem and encouraging the students to "try something." The authors observed pupils unobtrusively while they worked and took notes on the children's ways of working. After the class was finished, they collected the children's written work and examined it. The students' approaches to the problem fell into four categories.

The first group consisted of students who fell back on more or less random calculations, with or without a calculator. In this group, the students did computations that were far from the context of the problem. Examples from this group include 20 + 20, 6442, and so forth. That is, the children took the numbers from the original problem and computed with them without any attention, it would seem, to the conditions stated or the question that was originally asked. For example, one student wrote that "20 plus 64 is 84. So there are 42 parakeets and 42 elephants."

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The second group focused on one aspect of the problem, such as the number of heads or the number of legs, and reported the consequences as its final answer. For example, "There are ten parakeets and ten elephants," Mary said. "Why do you think you are right?" we asked. "Because there are twenty heads," she replied. Some other students thought that one elephant and nineteen parakeets were possible, because one plus nineteen is twenty. The fact that students in this group reported several different answers did not distress them, nor did the fact that their answers did not address the original conditions. If the computation was correct, students seemed to think that their answer was correct.

The third group of students could keep the information about legs and heads in mind simultaneously, but they coordinated it in unusual ways. Several students asked questions, such as, "Do all the critters have to have the normal amount of feet?" "Why do you ask?" we wondered. "Well, if you had ten parakeets and ten elephants, the parakeets would account for twenty legs. That would leave forty-four legs. So some of the elephants must have more than four legs."

The last group of students organized the information in the problem appropriately. Quite strikingly, these students almost always used a visual approach to solving the problem and used successive approximations, which we call "trial and success." Students represented the elephants and parakeets as shown in figures 38.1, 38.2, and 38.3.

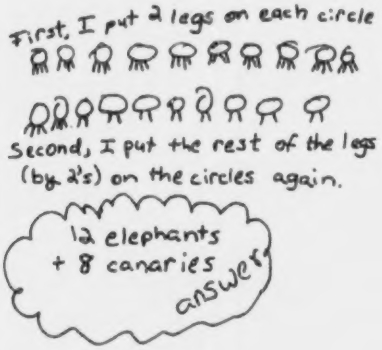
First, I put 2 legs on each circle
Second, I put the rest of the legs (by 2's) on the circles again.
12 elephants
+ 8 canaries
answer

Fig. 38.1. Starting with ten of each animal and adjusting legs to determine the correct answer

In figure 38.1, the children started with ten of each animal. Then they kept the total number of animals constant at twenty and adjusted the number of each type of animal to reach the correct total of sixty-four legs. The children who used the organizational approach shown in figure 38.2 counted the legs by ones, twos, or fours but rarely used their knowledge of multiplication. Figure 38.3 shows the work of a student who successfully combined the use of diagrams and knowledge of multiplication.

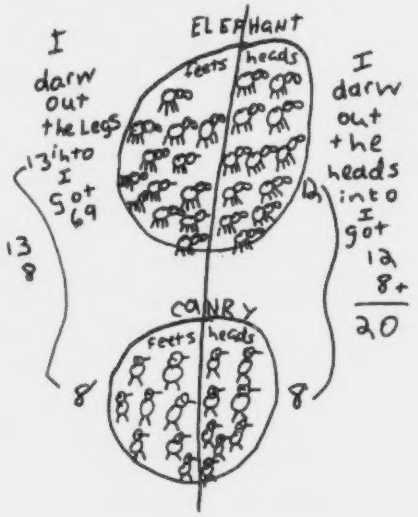
ELEPHANT
feets
heads
I darw out the Legs 13 into I got 69
13
8
8
I darw out the heads into I got
12
8+
20
12
CANRY
feets
heads
8

Fig. 38.2. Organizing animals to test a combination

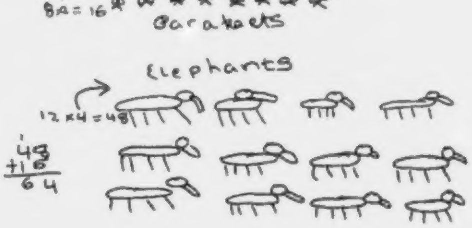
8×2=16
Parakeets
Elephants
12×4=48
48
+16
64

Fig. 38.3. Combining the use of diagrams with a knowledge of multiplication

Lesson 2: Finding and Applying Structure

In the next lesson, the elephant-and-parakeet problem was stated again and students discussed the approaches that they had used. Successful problem solvers shared their strategies. It had been two weeks since lesson 1, but students in both classes seemed very aware of the original problem, and we felt comfortable continuing with lesson 2. Students were now asked to pretend that they were making up problems for another class of fifth graders. The assignment was "Make up a problem just like the elephant and parakeet problem." The purpose of this assignment was to see whether students could derive the structure of the original problem.

The students' responses fell into four categories. The first category contained problems that seemed like complete nonsense. The second category of problems involved arithmetic. In the third category, several conditions were generated but the conditions did not involve logical necessity. The fourth category contained problems that were structurally the same as the original elephant-and-parakeet problem.

The nonsense problems, while logically inappropriate, were by far the most entertaining. One student wrote, "There was a zoo. The zookeeper checked the animals to see if they were still there every day. The guy came back and said there are altogether fifteen horses and fourteen cockatiels. The next guy comes back and says our count today is twelve legs and three heads. How many horses were sick? The zookeeper came and said there are six legs and twelve heads. There were fourteen cockatiels. The horses and cockatiels got sick. But they're better." Happy endings are important in fifth grade!

In the second category, the structure of the original problem was not reflected; for example, "A girl went to the grocery store. Everything cost three dollars. She got five eggs, two cartons of milk, three turkeys, and six packs of flour. How much money did she have?"

In the third category, several conditions were used, but the logical necessity of the original problem was missing. Our favorite example of this kind of problem was "A girl named Sandra went to McDonald's to get a cheeseburger. There are ten onions and eight pickles in each cheeseburger. There are fifty-six pickles and twenty onions. How many cheeseburgers would that make? If you can solve it, I'll give you that many cheeseburgers."

We did, however, have many youngsters who were able to construct a problem that was structurally the same as the original elephant-and-parakeet problem. One successful example of this task said, "There is a farm in Blugoland owned by the Spurdles. Mr. Spurdle went out and counted seven heads. Mrs. Spurdle went out and counted fifty-two legs. The Spurdles raised turtles and spiders. How many turtles and spiders are there?" Another farm problem asked the number of ducks and horses if twenty-seven heads and seventy-eight legs were counted. In yet another problem, two jungle experts were doing the counting. Only cheetahs and bald eagles lived in this jungle, and the counters found 100 heads and 350 feet.

Lesson 3: A Problem with Inconsistent Data

In the third lesson, a task similar to the original elephant-parakeet problem was posed, but the problem contained inconsistent data. That is, no whole-number solution existed. The new problem was as follows:

> Goochies and muckmucks live at the Strange Critter Hotel. Goochies have 8 legs and muckmucks have 5 legs. When the manager counted the legs of all twenty-four residents, all of whom were goochies and muckmucks, he got 127 legs. How many goochies? How many muckmucks?

Again, four categories of responses were noted. As with the original elephant-parakeet problem, some students simply conducted arithmetic operations without regard for how they might be related to the problem. One attempt at a solution was 8 × 5, as was 127 × 8. A third was "24 + 8 + 5 + 187 = 224. So there are 112 goochies and 112 muckmucks."

As in the original problem, many children seized on one aspect of the situation and reported its consequences as a final answer, for example, "There are sixteen goochies and eight muckmucks because sixteen plus eight is twenty-four."

A larger group of students seemed to understand the structure of the problem this time around. This group almost always used drawings. As before, they often counted by ones, fives, or eights instead of using multiplication. Commonly, they would posit an answer—for example, twelve goochies and twelve muckmucks, which fit the condition of twenty-four residents in all—and then would distribute legs to the goochies or muckmucks. When they could not get the number of available legs to work, some students wanted defective goochies or muckmucks.

A smaller group of children claimed that no answer was possible. One child explained, "Well, 127 can't be divided by 8. And it's not divisible by 5. And even if you add 8 and 5 and try to divide it, it doesn't work. Thirteen doesn't go into 127. So, it's impossible." Even though several students recognized that the problem could not be solved with whole numbers, only one of the more than forty students could show this restriction to be true. This student showed all possible combinations from one goochie and twenty-three muckmucks to twenty-three goochies and one muckmuck and counted the number of legs for each combination.

Implications for the Curriculum

As noted earlier, many of the students in these two fifth-grade classes were eager problem solvers and enjoyed the thinking and problem solving, even if they were not initially correct in their solutions to the problems. This kind of activity gave the students lots of opportunity to talk with one another, to try different ideas without worrying about a grade for the gradebook, and to generate and refine problem-solving strategies. Give students the chance to deal with unusual problems, problems for which they have not been taught specific methods.

The trial-and-success method of altering critters' legs worked very well for the students. One major implication for the curriculum is to place more emphasis on trial-and-success. Try something, learn its consequences, and refine your approach for another try. This tactic is fairly important in real life as well as in mathematics.

It seems reasonable to question the effectiveness of arithmetic drill as the main course in children's mathematical diets. These students had had four years of drill and practice in arithmetic, and many of them still did not make thoughtful decisions about how to use arithmetic in their problem solving. Many students performed random computations unrelated to the task. And virtually all the students who did engage the task successfully did not perform computations but, rather, drew critters and counted legs. Teachers and curriculum-makers should keep in mind that arithmetic procedures taught out of context, as in a textbook drill sheet, do not automatically generalize beyond that drill page. Although the newer textbooks often contain pages that relate to specific problem-solving strategies, these problem-solving activities do not occur often enough.

The proportion of students who resorted to visual rather than numerical methods of solution was striking. Is it possible that students have strong visual abilities that are underaddressed by the present curriculum? Teachers and curriculum-makers should keep in mind that visual approaches should get far greater emphasis in teaching and learning mathematics.

The students' inability to show that data are inconsistent, as in lesson 3, may suggest that inconsistency should be covered by the mathematics curriculum. Real-world data are often inconsistent, and not every problem has one and only one solution. Teachers and curriculum-makers should keep in mind that unorthodox problems of the type used here might be a welcome and productive challenge to children's thinking, lest children get the impression in mathematics class that all problems are tidy and instantly solvable.

Perhaps the most important finding is that many of these fifth-grade students had difficulty keeping several things in mind at once. We noted that difficulty in many of the students who answered "ten elephants and ten parakeets" and let the information about legs drop out completely. We think that this arena is worth exploring from a research point of view. Is it simply a function of these particular children and these particular problems? We think not, because we have seen it in several fifth-grade classes on a wide variety of mathematical tasks. If keeping in mind several things at once is an aspect of thinking that is underdeveloped in fifth graders, we think that all curricula—not just in mathematics—should address this issue.

Implications for Classroom Teaching

Although this article addressed specific activities given to two groups of fifth graders, we think that simply taking the time to approach problem solving in a small-group format does a lot to raise the confidence of the students in their ability to solve problems. Whether you use goochies and muckmucks or tangrams or tessellations, taking some time to work together, communicate, and de-emphasize rote arithmetic is important in the curriculum.

A second consideration for teachers: Do you want students to draw their solutions all the time? Probably not, but you do want them to know that drawing is acceptable if it helps their problem-solving capability. We should encourage children to draw more frequently if that solution strategy is what makes sense to them.

One of the hardest things to do as a classroom teacher is to change the way that things have been done for years. Although most textbooks offer pages of word problems, which are often not really problems but are computational exercises couched in words, much richer activities can be used in the classroom. We recommend that you try the farm-animal problems here or make up some similar problems.

Fair warning: Once you start problem solving with students, it is hard to stop. The students enjoy thinking, and they enjoy being challenged if the challenges are within their range. What is the implication here? Clearly one has to learn about one's students—learn what is a good stretch, what is an overmatch, and what is an undermatch. Problems of the sort discussed in this article were very helpful in giving us a fresh look at children's abilities. We learned very quickly which children were locked into random computations, which children captured the essence of a complex problem, and which children could generate their own solution strategies. We hope that you will consider using nonroutine problems to gain insight into children's ways of thinking.

From a classroom-teaching point of view, we should never forget that mathematics is fundamentally a sense-making activity. This precept is all too easy to forget when a great deal of our time is spent imparting adult-grown facts, rules, and procedures, often with little or no sense being made by students. We want students to be able to make sense of situations and not blindly copy what a teacher says. We need to give our students interesting situations to make sense of, and we must ask questions to challenge them to go further in terms of sense-making and generalizability of ideas, principles, and strategies.

State-ing the Facts: Exploring the United States

by Jennifer M. Bay, Ann M. Bledsoe, and Robert E. Reys

THE DIVERSITY of our fifty states is known by most Americans; less acclaimed is the fact that the fifty states can form the background for our students to explore some important mathematics. The following progression of activities includes such important mathematics topics as estimation, scaling, area of nonstandard shapes, algebraic thinking, and real-life situations for squaring and square roots. These activities allow the use of technology by employing such geometry software packages as The Geometer's Sketchpad (Jackiw 1995), Cabri (Texas Instruments 1997), and Geometric Supposer (Sunburst Communications 1997). For us, the activities offered an excellent integration of mathematics and social studies, one that our students enthusiastically enjoyed!

Many of us do not have a very good idea of how big large areas are. For example, how large is a 220-acre farm? How does it compare with a city block? Did you know that the area of Yellowstone Park is 3,472 square miles? A forest fire in California burned 18,924 acres. How would these areas compare with the size of *your* state? What does measuring in square units mean? To help them answer these questions, our students need a benchmark, a referent that has meaning for them that will aid in their ability to estimate other quantities. For example, knowing our height helps us estimate how tall other objects are. But what referents do we have for area?

If we take some time to explore the area of our own state, we can use it as a benchmark to help us understand real-world situations. We can also compare it with the areas of other states, regions, and countries. A map of the United States can be helpful in providing gross visual comparisons. However, maps can be deceptive and are sometimes difficult to interpret, since the shape of a state is irregular. A more commonly recognized geometric figure, like a square or rectangle, can help. Figure 39.1 illustrates how The Geometer's Sketchpad can transform an unusual shape—the border of Missouri—into a square, with each successive figure having the same area.

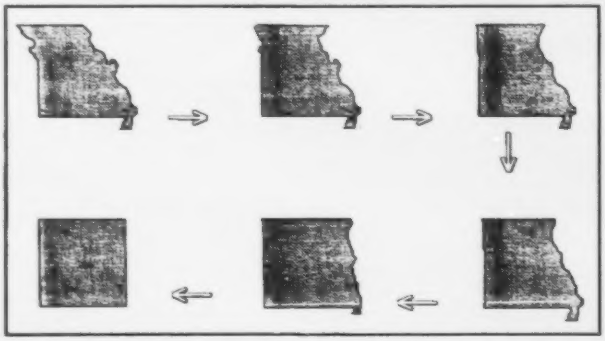

Fig. 39.1 A metamorphosis of a map of Missouri as it moves toward a square

We created Missouri by placing many points in the approximate shape of its border. We then connected these dots with segments. Once we had created a polygon, we had the area displayed on the screen. We could then drag and stretch any segment and ensure that the resulting figure retained the same area. As students watched this process, they saw that geometric figures can look very different but have the same area.

This demonstration launched our class into a great eight-day, eighth-grade class project: an interdisciplinary unit on the United States. In mathematics, our goal was to explore ways to compare oddly shaped areas. We

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began on a small scale: comparing the areas of each student's hand and foot with those of classmates. Class began with a discussion of who in the class had the biggest and smallest hands. Much debate ensued, with such hypotheses as that the taller students would have the larger hands and that people with long fingers would have a bigger hand area. Each student traced her or his hand on grid paper, estimated its area, and then tried to find the area by counting squares. We compared our findings; although some hands were obviously bigger than others, some comparisons were not easy because of the different shape of each hand. With some discussion, we agreed that a good way to compare our hands would be to re-form them into one common shape. We decided on the square. Each student cut out his or her hand tracing, then cut it apart and fit it together to make a square. This task was complicated by the students' desire to discard partial pieces of the grid squares and by their difficulty with fitting together the tiny pieces. As we hung our resulting squares on the wall, we could *see* the size differences. We duplicated this process with feet. The results were surprising: the areas of a student's hand or foot did not necessarily relate to his or her height or age.

We then made a sizable jump in our study of areas: the fifty states. To increase their curiosity and to practice their estimation skills, students worked at their tables in groups of four or five to number the states from largest to smallest, using only a map as a reference. This task certainly became an interesting geography lesson for us, since we needed to know which state was where. Before obtaining more precise areas for the states, we supplied the area of each of a few states that made good benchmarks: Colorado (103,595 square miles), Vermont (9,273 square miles), and Rhode Island (1,210 square miles), which are near the useful benchmarks of 100,000, 10,000, and 1,000. When students checked their results with an almanac, they were often surprised, even amazed, at how the "rankings" fell; for example, they were sure from looking at the map that Tennessee was bigger than Louisiana. Table 39.1 summarizes facts about the states that students reported.

To familiarize students with the states and to incorporate some algebra, we challenged them to use the areas of the states and record patterns or relationships that they observed. In teams, students brainstormed and produced lists like the one that follows, recording the English and the mathematical sentences:

Nevada is about ten times the size of Maryland.
(NV = 10(MD).)
Florida and Georgia are about the same size.
(FL = GA.)
Alaska is a bit larger than Texas, Oklahoma, New Mexico, and Colorado combined!
(AK = TX + OK + NM + CO.)
Five Delawares would equal about two Connecticuts, which is about the size of Vermont.
(5(DE) = 2(CT) = VT.)
New York is smaller than Colorado. (NY < CO.)

We discussed the difference between the symbol for equals (=) and approximately equals (≈) and what size difference would qualify as approximately equal. Encouraging students to create other comparisons is a natural way to extend and promote creative, proportional, *and* algebraic thinking.

How could we better envision the size of the states? Because of students' prior experience with hands and feet, we were not surprised when a student said, "Let's make 'em squares!" "We cannot cut out all those states and put 'em together like we did our hands, it'd take too long!" responded another student. Certainly cutting out each state and recomposing it into a square is quite labor intensive. We asked, "Is there another way we could make states into squares?" "We could 'guess and check' with the area," someone volunteered.

Our students began with Alaska, guessing what the side of the square must be to give a matching area. They revised their estimates until they had narrowed in on a side length. Without knowing it, students were practicing the estimation of squares and square roots. They estimated how many miles a side would be and then checked their guesses on the calculator. Estimates varied widely, since several students found the side by dividing the area by 4. "No, that doesn't work," argued a student when this strategy was shared. "Dividing by four doesn't work because you are not finding area by timesing by four, it's by the number, so you have to do it a different way." Students convinced one another how they must guess the side of a square, given its area. We asked students to use their calculators, enter the area of the state, and find the square root. Many students realized that it would be the same as the length of the side of the square, but for others, this connection was new—a link between actual squares and the terminology *square* and *square root*. "Squaring a state" gives a real-world context for students to establish these connections between geometry and arithmetic.

It was time to tackle the main task—converting all the states to squares to give us a visual representation that

Table 39.1
United States Fact Summary

State	Area (Sq. Miles)	Rank in Area	Population	Rank in Population	Capital
Alabama	51,609	29	4,252,982	22	Montgomery
Alaska	586,412	1	603,617	48	Juneau
Arizona	113,909	6	4,217,940	23	Phoenix
Arkansas	53,104	27	2,483,769	33	Little Rock
California	158,693	3	31,589,153	1	Sacramento
Colorado	104,247	8	3,746,585	25	Denver
Connecticut	5,009	48	3,274,662	28	Hartford
Delaware	2,057	49	717,197	46	Dover
Florida	58,560	22	14,165,570	4	Tallahassee
Georgia	58,876	21	7,200,882	10	Atlanta
Hawaii	6,450	47	1,186,815	40	Honolulu
Idaho	83,557	13	1,163,261	41	Boise
Illinois	56,400	24	11,829,940	6	Springfield
Indiana	36,291	38	5,803,471	14	Indianapolis
Iowa	56,290	25	2,841,764	30	Des Moines
Kansas	82,264	14	2,565,328	32	Topeka
Kentucky	40,395	37	3,860,219	24	Frankfort
Louisiana	48,523	31	4,342,334	21	Baton Rouge
Maine	33,215	39	1,241,382	39	Augusta
Maryland	10,577	42	5,042,438	19	Annapolis
Massachusetts	8,257	45	6,073,550	13	Boston
Michigan	58,216	23	9,549,353	8	Lansing
Minnesota	84,068	12	4,609,548	20	Saint Paul
Mississippi	47,716	32	2,697,243	31	Jackson
Missouri	69,686	19	5,323,523	16	Jefferson City
Montana	147,138	4	870,281	44	Helena
Nebraska	77,227	15	1,637,112	37	Lincoln
Nevada	110,540	7	1,530,108	38	Carson City
New Hampshire	9,304	44	1,148,253	42	Concord
New Jersey	7,836	46	7,945,298	9	Trenton
New Mexico	121,666	5	1,685,401	36	Santa Fe
New York	49,576	30	18,136,081	3	Albany
North Carolina	52,586	28	7,195,138	11	Raleigh
North Dakota	70,665	17	641,367	47	Bismarck
Ohio	41,222	35	11,150,506	7	Columbus
Oklahoma	69,919	18	3,277,687	27	Oklahoma City
Oregon	96,981	10	3,140,585	29	Salem
Pennsylvania	45,333	33	12,071,842	5	Harrisburg
Rhode Island	1,214	50	989,794	43	Providence
South Carolina	31,055	40	3,673,287	26	Columbia
South Dakota	77,047	16	729,034	45	Pierre
Tennessee	42,244	34	5,256,051	17	Nashville
Texas	267,338	2	18,723,991	2	Austin
Utah	84,916	11	1,951,408	34	Salt Lake City
Vermont	9,609	43	584,771	49	Montpelier
Virginia	40,817	36	6,618,358	12	Richmond
Washington	68,192	20	5,430,940	15	Olympia
West Virginia	24,181	41	1,828,140	35	Charleston
Wisconsin	56,154	26	5,122,871	18	Madison
Wyoming	97,914	9	480,184	50	Cheyenne

would allow us to compare the sizes of states. Determining what scale to use, to our surprise, became an exciting, meaningful activity in and of itself. As a class, students brainstormed possible scales (i.e., 1 cm = 1 mile, 1 m = 1 mile, 1 in. = 1000 miles, 1 in. = 100 miles). In groups, students were to find the best scale that would work and meet two rules: (1) Alaska had to fit on one piece of twelve-inch-by-eighteen-inch construction paper, and (2) Rhode Island had to be big enough to see. This difficult and thought-provoking task was approached in many ways.

One group let 1 millimeter equal 50 miles. When asked to tell how big Alaska would be, they noticed that 1 centimeter would be 500 miles, which was definitely too small. They knew that they needed to adjust, but they were not sure what to adjust. After much exploration, the students presented their findings, defending why they thought that their scale would be the best:

> 1 inch = 78.6 miles. This makes Alaska exactly a 10 by 10 [inch] square.
>
> 1 inch = 100 miles. Alaska will fit on the page, and Rhode Island is still big enough to cut out—it's a little less than 1/2 an inch on each side.
>
> 1 inch = 64.4 miles. This makes Alaska fit exactly on the construction paper, so Rhode Island will be as big as it can be, too.

This last statement sparked a surprising discussion. One student said. "That number is too messy," and someone else said, "Let's round it to 64." The teacher intervened and said, "Let's see if 1 inch equals 64 miles works." Students punched numbers into their calculators and started shaking their heads. One student said, "It's too big, try 60." So they did. "Hey, we're going the wrong direction," someone said. Another responded, "Yeah, round to 65." As teachers and mathematics educators, we were excited to see the need for flexible rounding. In the end, students agreed that 1 inch equals 100 miles was the scale they wanted to use, since 100 was a "nicer" number.

Groups were assigned different states. Each group measured squares for its states and cut them out. As students began hanging their results and ordering them on the wall, they noticed that some states that were supposed to be bigger, according to the actual area, had smaller squares. Two problems had occurred during the measuring: students had started measuring at the end of the ruler, instead of at zero, and, more commonly, they could not convert the decimal given by the calculator to the corresponding fraction of an inch. This discrepancy encouraged all students to check their calculations. If students got a square with a side of 7.2 inches after taking the square root, they had to estimate what fraction of an inch 0.2 inch was. This estimation was not hard after it occurred to them that 0.2 on the calculator did not mean two marks on the ruler. In the end, the students lined up the "squared states" and had an excellent visual display of how the fifty states compared in area (see fig. 39.2).

Photograph by Jennifer Bay; all rights reserved

Fig.39.2. A visual comparison is made.

The foregoing activity sparked students' curiosity about how the populations would compare. Does the largest state have the largest population? How would the states be ordered by population? What would a visual representation look like? The students decided to attack the problem in a similar fashion: create a scale and make squares to represent each state's population, which was taken from table 39.1.

The Geometer's Sketchpad was also used to create the squares. Although this technology is powerful, it still requires careful measures and conversions to ensure that the desired products—here, squares of appropriate sizes—result. See figure 39.3 for several hints on constructing squares with Sketchpad.

An illustration of the power of Sketchpad is seen in examining the squares of a few states. We used a scale of 1 inch to 100 miles for Texas, which gives a square that fits nicely on a page (fig. 39.4). Figure 39.5 shows the results from several other states. One advantage of

Constructing a Square with The Geometer's Sketchpad

The construction of a square requires more than simply plotting four points and connecting the dots. Some necessary skills are knowing how to construct a segment, a perpendicular line, and a circle with a given center and radius. The arcs of the circles are used to construct segments of equal length. Once a square has been constructed, measure the length of the side and then drag one of the vertices to create a square with the appropriate side length. Students can then measure the area of the figure and observe the relationship between the length of the side and the area of the corresponding square. As the vertex is dragged, the changes in the length of a side and the corresponding area can be displayed in dynamic fashion on the screen.

Fig. 39.3. Constructing a square with The Geometer's Sketchpad

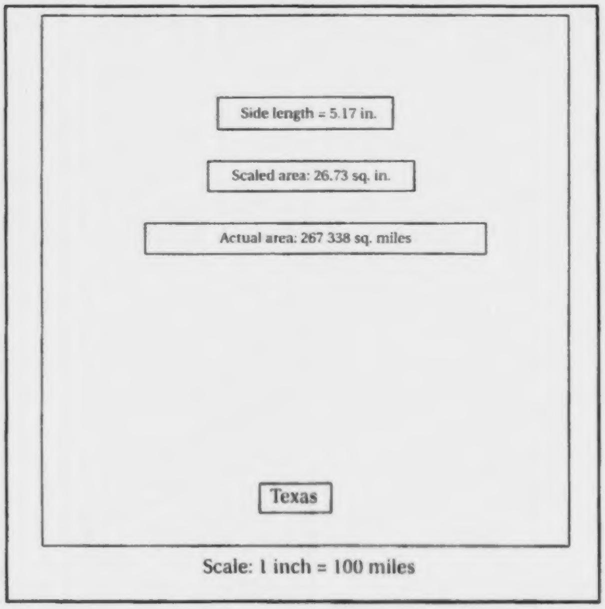

Side length = 5.17 in.
Scaled area: 26.73 sq. in.
Actual area: 267 338 sq. miles
Texas
Scale: 1 inch = 100 miles

Fig. 39.4. A square of Texas

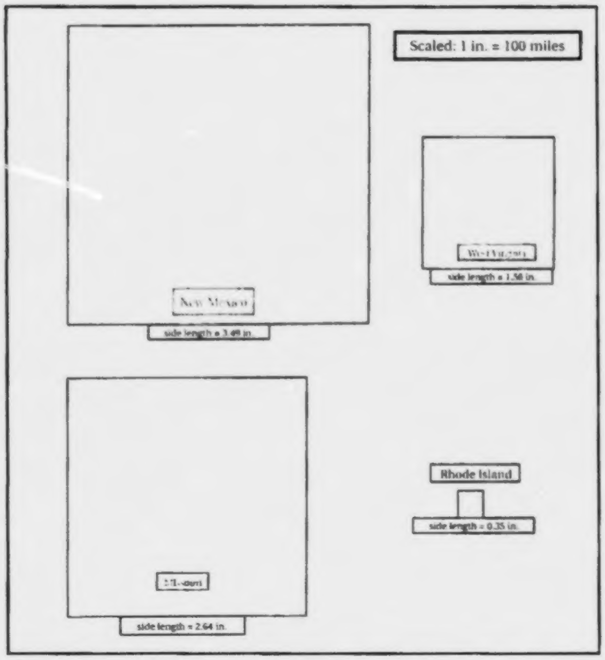

Scaled: 1 in. = 100 miles
New Mexico
side length = 3.49 in.
West Virginia
side length = 1.58 in.
Missouri
side length = 2.64 in.
Rhode Island
side length = 0.35 in.

Fig.39.5. A view of the squares of some states

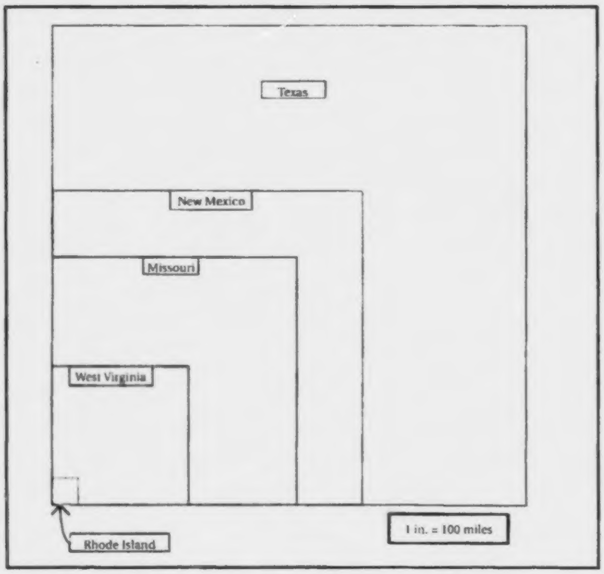

Texas
New Mexico
Missouri
West Virginia
Rhode Island
1 in. = 100 miles

Fig.39.6. Another perspective of the squares of the same states

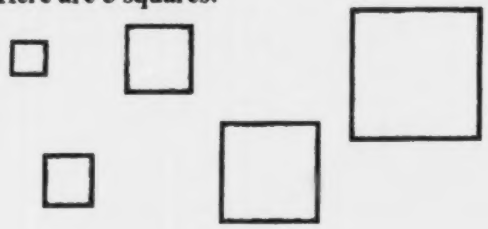
using Sketchpad is that, once the squares are created, they can be superimposed on one another, which gives a good visual for size comparison (fig. 39.6).

Whether we used construction paper or The Geometer's Sketchpad, after completing the squares we returned to the original lists of comparisons compiled by the students. For example, if students wrote that it would take 200 Rhode Islands to equal one Texas, did their original "equations" and the size of their squares agree? More questions can be generated, such as, "How do the sides of the squares compare with their areas?"

In addition to being exposed to all the mathematics woven into this exploration, students learned much about their country and social issues. For example, the social studies teacher took the opportunity to discuss population and the electoral college with the students. Within our class, students' questions led to an integration of other topics. When we looked at population density, we discussed using a scale, such as one dot for every 100 000 people, and putting the correct number of dots on each "state square." Which states would be crowded with dots? Which would have only a few dots? The discussion of population density led to discussions about the problems associated with highly populated states versus the problems of less populated states. Questions about topology arose: Is the area of a mountainous state, like Utah, figured by the amount of land in the state, that is, its surface area? If so, could a flatter state of a slightly smaller area, like Kansas, actually take up more space on the map? Geography was clearly integrated throughout. Further opportunities for extending these ideas are limitless. Puzzles or challenges, like the world scavenger hunt in figure 39.7, can be presented.

WORLD SCAVENGER HUNT

Here are 5 squares:



Match them to these regions:

North America
Asia
Pacific Ocean
Europe
Atlantic Ocean

Key: North America: 9,400,000 square miles
Asia: 17,000,000 square miles
Pacific Ocean: 64,000,000 square miles
Europe: 3,800,000 square miles
Atlantic Ocean: 32,000,000 square miles

Fig. 39.7. Example of a puzzle comparing regions to squares

Students could represent the areas of the states with other geometric shapes, such as circles or equilateral triangles. They could create rectangles having a common side length. Presented in this way, the states appear to form a bar graph once they are posted next to one another. Our favorite extension, created by our students, was making a square to represent the area of the United States and trying to fit all the "state squares" inside it.

Yet another option is a project in three dimensions: "Cubing the Salt Lake." We asked students a question: "What would be the dimensions of a cube large enough to hold the water of the Salt Lake?" Using dot paper or cubes or both, students can explore the relationships of sides to volume—cube and cube root—and create other visual representations. This activity not only helps them develop benchmarks in three dimensions but gives them experience with isometric drawings. One of the powers of mathematics is that solving one problem stimulates generalizations and thinking that lead to more interesting and challenging activities.

Initially we wanted students to develop a meaningful understanding of large areas and wanted to help students develop some meaningful benchmarks. Our project, however, led to more than a week of exploration and the study of many additional important mathematical ideas. As teachers, we learned how much can be gained from presenting a real problem. Students had to measure accurately; they had to determine reasonable scales and agree on them; and they needed to be careful with their rounding when using the calculator, or their results—the visual display comparing states—would not make sense. Although these skills were not the focus of the activity, we believe that giving students an opportunity to undertake a task that requires such exactness is important. The activity stimulated students to think carefully about their work, interact cooperatively, debate among themselves, and reason mathematically. All these benefits, *plus* the wide range of mathematics and the ties to other content areas, made this activity exciting and meaningful and one that we shall certainly continue to use and adapt for future middle school classes.

References

Jackiw, Nicholas. The Geometer's Sketchpad. Berkeley, Calif.: Key Curriculum Press, 1995. Software.

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ACTIVITIES

Problem Solving and Representation

Rachlin: "Learning to See the Wind"

1. Find several different ways to solve problem 3 in figure 36.1. Alternatively, observe the strategies used by a group of students to solve this problem. *Group activity*: Discuss how a comparison of these strategies can be used to guide a class discussion on the distributive property. Write a report of your group's activity.

2. *Group activity*: Find a solution to the question posed in figure 36.4, and share your strategy with your group. As a group, discuss the different strategies, thought processes, and understandings that were used.
   Individual report: Write out your solution to the problem. Discuss any insights that you gained from the group discussion of the different solutions.

Kelly: "Improving Problem Solving through Drawings"

Select a problem for which you found it helpful to draw a picture as you solved it.

a) State the problem, and describe your solution strategy.
b) Discuss how your drawing helped you understand the problem and reach a solution.

Bey, Bledsoe, and Reys: "State-ing the Facts: Exploring the United States"

Identify three different representations that the students used to compare the relative areas of different states.

a) Discuss what information was lost and what was gained by using each form of representation.
b) Which form of representation do you prefer for such a comparison, and why?

Section 1: Problem Solving and Representation

The discussion of the Representation Standard in *Principles and Standards* (NCTM 2000, pp. 67–71) includes the following statements:

- Representation can help make mathematical ideas more concrete and available for reflection.
- Teachers can gain valuable insights into students' ways of interpreting and thinking about mathematics by looking at their representations.
- Different representations often illuminate different aspects of a complex concept or relationship.

Select one of these statements, and discuss in detail how it is illustrated by one or both articles in this section. Be specific, and include examples from the articles.

2.

COMMUNICATION; REASONING AND PROOF

COMMUNICATION, and reasoning and proof are essential to the process of learning and doing mathematics, as is evident in the ways that these Process Standards are found in most of the articles contained in this volume. The articles included in this section give examples of students engaged in productive mathematical communication and illustrate how justification and proof are important aspects of this process.

Communication

INSTRUCTIONAL programs from prekindergarten through grade 12 should enable all students to —

- *organize and consolidate their mathematical thinking through communication;*
- *communicate their mathematical thinking coherently and clearly to peers, teachers, and others;*
- *analyze and evaluate the mathematical thinking and strategies of others;*
- *use the language of mathematics to express mathematical ideas precisely.*

The article by McGrath illustrates how important communication skills can be introduced and practiced using graded group problem-solving sessions. Forms for student self-assessment and teacher evaluation are used to help students recognize and improve such skills as listening, asking clarifying questions, recognizing and communicating strategies, and facilitating group involvement.

Rubenstein discusses the roots, meanings, and words related to various mathematical terms. She explains how understanding word origins can help middle school students make sense of and appropriately use the language of mathematics. Incorporating language development into mathematics instruction helps verbal learners as well as English–second language students gain better access to mathematics.

What Is the Score on Scored Discussions?

by Kathleen L. McGrath

I believe that humans do not exist in a vacuum and that one of my most important jobs as a teacher is to foster and support the progress of my students as lifelong learners who are productive, successful members of groups working on a common goal or problem. Cooperative learning is a wonderful tool for this purpose.

I have tried various techniques to implement cooperative learning with success and was eager to try Leach's (1992) scored discussions with my seventh graders. This technique offered a way for teachers and students to model, practice, and internalize many valuable cooperative-learning behaviors in the classroom. My students work well in cooperative groups, but I always look for ways to improve the interaction. The more my students work in a group in a focused, thoughtful, and supportive way, the more they are able to learn.

In cooperative groups, learning can be more than the sum of the parts (Mousley, Clements, and Ellerton 1992). Because more students are actually engaged at any one time, more ideas, critical thinking, and intellectual risk taking occur in cooperative groups than in whole-class experiences (Davidson 1990; De Jong and Hawley 1995; Karre 1995). Moreover, data sharing, brainstorming, and problem solving occur in cooperative groups (Davidson 1990) at greater levels than when individuals work alone—ideas from one student are learned by others and trigger new ideas.

KATHLEEN McGRATH teaches at Edgewood Middle School, Highland Park, IL 60035. Two beliefs guide her teaching: everyone is smart, and we can all benefit from one another's talents as we learn. Cooperative learning is an excellent way to enact those beliefs.

What Makes Cooperative Groups Work?

Certain behaviors are present in cooperative groups that work successfully. These behaviors include soliciting opinions, encouraging explicitness, pinpointing differences, interrelating viewpoints, being willing to give mutual support, sharing ideas and information, encouraging and praising fellow members, checking that everyone in the group understands (Cohen 1994), and speaking face to face (Strother 1990; Sutton 1992). Most of these behaviors do not occur spontaneously—students need training to develop them (Cohen 1994; Sutton 1992). This article reports on my attempts to implement Leach's (1992) technique of a scored discussion to see if it might be valuable in training students to learn and improve their behaviors that are important for group success.

What Is a Scored Discussion?

For my purposes, a scored discussion is a five-minute session involving three to four students who are scored by a teacher in relation to a rubric while they work on solving a problem. They may either be working in front of the whole class or as one group while all students are in a group.

Leach's (1992) scored-discussion rubric that follows listed most of the behaviors previously mentioned in research findings. In my scoring system, items in parentheses are points assigned to behavior and starred items are additions I made to cover other desirable behaviors. The following are positive behaviors:

1. Determining a possible strategy to use (3)
2. Communicating a strategy successfully (3)
3. Applying a property correctly (2)
4. Recognizing misused properties or arithmetic errors (2)

5. Drawing another person into the discussion (2)
6. Asking a clarifying question (3)
7. Moving the discussion along (1)
8. Encouraging or praising partner (2)*
9. Making sure that everyone understands (2)*

The following are negative behaviors:

1. Not paying attention or distracting others (–2)
2. Interrupting (–2)
3. Making an incorrect application or assumption (–1)
4. Monopolizing (–3)
5. Making a personal attack (–3)

The Setting

I taught three regular seventh-grade mathematics classes, averaging twenty-three students, with ten special-education students distributed among the three classes, and one prealgebra class of twenty-five seventh graders. The scored discussion began midyear, after the second quarter started. All my classes participated almost daily in some type of cooperative learning, either as partners or in small groups, changing partners or groups about every four to six weeks. Generally, the students had become better at working together than at the year's start.

Introduction to the Students

After explaining those behaviors that were important for members to exhibit or avoid for cooperative groups to be effective, I distributed my modified list of behaviors (fig. 40.1). As a class we discussed each behavior on the self-assessment sheet (fig. 40.1), including examples.

Next, I asked the students to think how they had worked in cooperative groups previously. I asked them to use their past experiences to fill in the individual self-assessment sheet, using the given scale: 1—never, 2—seldom, 3—sometimes, and 4—often. Then they

Self-Assessment

Name ______________________________
Final total ___________________________

Please choose the best description of you for each question and circle the corresponding number.

Behaviors in a group/partnership	Often (4)	Sometimes (3)	Seldom (2)	Never (1)
1. Determining a possible strategy to use	4	3	2	1
2. Successfully communicating a strategy	4	3	2	1
3. Correctly applying a property	4	3	2	1
4. Recognizing misused properties or arithmetic errors	4	3	2	1
5. Drawing another person into the discussion	4	3	2	1
6. Asking a clarifying question	4	3	2	1
7. Moving the discussion along	4	3	2	1
8. Encouraging or praising a partner	4	3	2	1
9. Making sure everyone understands	4	3	2	1

Total A = ____________

1. Not paying attention or distracting others	4	3	2	1
2. Interrupting	4	3	2	1
3. Making an incorrect application/assumption	4	3	2	1
4. Monopolizing	4	3	2	1
5. Making a personal attack	4	3	2	1

Total B = ____________
Total A = ____________
– Total B = ____________
Final total = ____________

Fig.40.1. Self-assessment form

tallied the scores on their filled-out sheets, adding the points for positive behaviors and subtracting the points for negative behaviors. This scoring method had three purposes: (1) It allowed students to think about themselves in relation to the positive and negative behaviors before experiencing the scored-discussion procedure, (2) it gave me an idea of where my students saw themselves, and (3) it familiarized students with the behaviors that would be observed in the scored discussions.

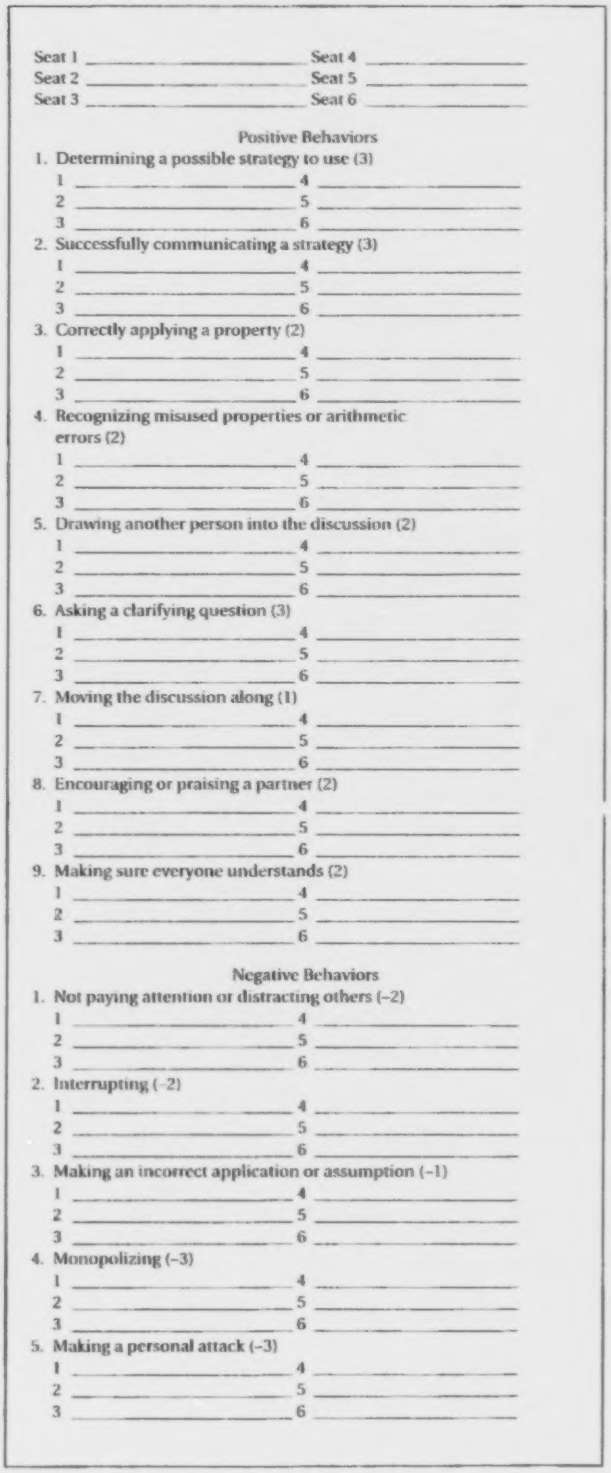
The results in all my classes were consistent. The mean, median, and mode of all the student assessment sheets were about 21. Dividing 21 by the number of positive behaviors gave an average of little more than +2.3. Thus, most students saw themselves, on average, as exhibiting positive group behaviors a little more than seldom and less than often, which seemed consistent with my observations. After these results were in, I explained that every student would participate in a practice scored discussion in front of the class in a group of three or four over the next few days to experience how this process works.

Scored Discussion—Practice Round

Two girls and two boys volunteered to be the first group in front of the class. I used the rubric form (fig. 40.2), making a check mark each time a student demonstrated one of the behaviors on the rubric.

Desks were placed in front of class about three feet apart so that students would have to speak audibly. The seating pattern was boy-girl-boy-girl. Another student volunteered to be the scribe, whose job was to record the work on the chalkboard so that we could more easily follow the logic and discussion. This first scored-discussion group worked on a surface-area problem (fig. 40.3).

For each scored-discussion group in front of the class, I distributed to the rest of the students an individual grading sheet with points on it (fig. 40.4). Each student who was not in the scored-discussion group chose one student to observe who was in the volunteer scored-discussion group. The observers made checks each time they observed a behavior on the sheet (fig. 40.4). At the end of the scored discussion, students who marked a rubric would tally the final score, giving points to each check mark. For example, if two checks were at "determining a possible strategy to use," then



Seat 1 ____________ Seat 4 ____________
Seat 2 ____________ Seat 5 ____________
Seat 3 ____________ Seat 6 ____________

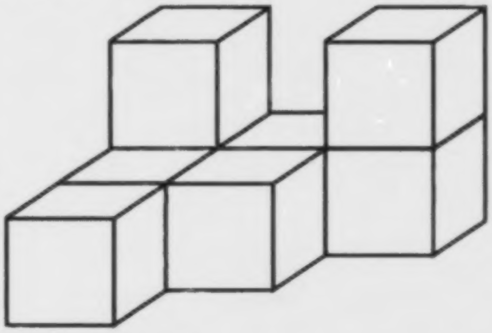
Positive Behaviors

1. Determining a possible strategy to use (3)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
2. Successfully communicating a strategy (3)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
3. Correctly applying a property (2)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
4. Recognizing misused properties or arithmetic errors (2)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
5. Drawing another person into the discussion (2)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
6. Asking a clarifying question (3)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
7. Moving the discussion along (1)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
8. Encouraging or praising a partner (2)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
9. Making sure everyone understands (2)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________

Negative Behaviors

1. Not paying attention or distracting others (−2)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
2. Interrupting (−2)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
3. Making an incorrect application or assumption (−1)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
4. Monopolizing (−3)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________
5. Making a personal attack (−3)
1 ____________ 4 ____________
2 ____________ 5 ____________
3 ____________ 6 ____________

Fig.40.2. Rubric form for the scored discussion to up to six students



Find the surface area.

Fig. 40.3. The problem used in the first scored discussion

six points were given. The practice scored discussions were conducted for instructional purposes, so these rubric grades were not used in determining the student's mathematics grade.

Participating in whole-class scored discussions as a group member or a rubric scorer served two purposes. First, it further familiarized the students with the behaviors on the rubric. Second, scoring a fellow student helped students spot how these behaviors would appear in practice. The audience enjoyed this aspect. When they were not participating in the scored-discussion groups, students continued to observe and mark the rubric for a student in the groups in front of the class. We agreed not to share comments in class, as that approach was reported to be too uncomfortable for the students in the scored discussion. To provide feedback, I asked that the scorers give their tallied sheets to the person they observed. After the first scored-discussion trial, I divided the class so that each scored-discussion participant would have about the same number of rubric sheets for feedback. In all four classes, every scored-discussion participant talked, some more than others. Students offered possible strategies with applications, checking for understanding and correctness with group members along the way.

Individual Evaluation Form

Name ____________________ Date ____________ Total ____________

Positive

1. Determining a possible strategy to use (3) ____________
2. Successfully communicating a strategy (3) ____________
3. Correctly applying a property (2) ____________
4. Recognizing misused properties or arithmetic errors (2) ____________
5. Drawing another person into discussion (2) ____________
6. Asking a clarifying question (3) ____________
7. Moving the discussion along (1) ____________
8. Encouraging or praising a partner (2) ____________
9. Making sure everyone understands (2) ____________

Positive Total ____________

Negative

1. Not paying attention or distracting others (–2) ____________
2. Interrupting (–2) ____________
3. Making an incorrect application/assumption (–1) ____________
4. Monopolizing (–3) ____________
5. Making a personal attack (–3) ____________

Negative total ____________
Final total = ____________

Fig. 40.4. Scoring sheet for use by student observers

The Continued Process

After the first whole-class scored discussion, I divided the remaining students into groups and continued with one or two scored discussions in front of class each day in each class. The problems used during this time were review (figs. 40.5 and 40.6) from a unit on area and volume and difficult problems from previous or upcoming homework and from a mathematics contest. After two days of whole-class scored discussion, I asked students to respond in their journals to the following questions:

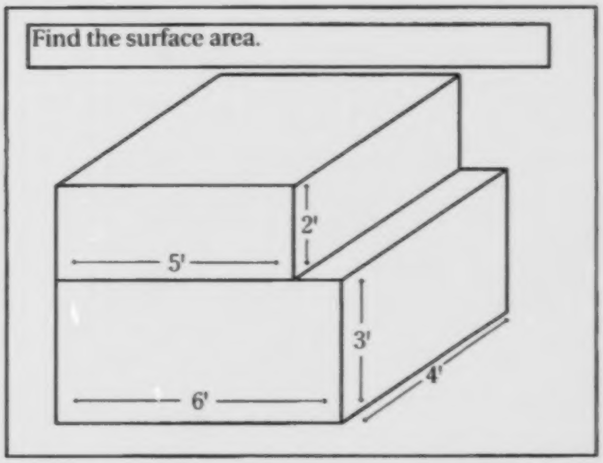


Fig. 40.5. Another scored-discussion problem

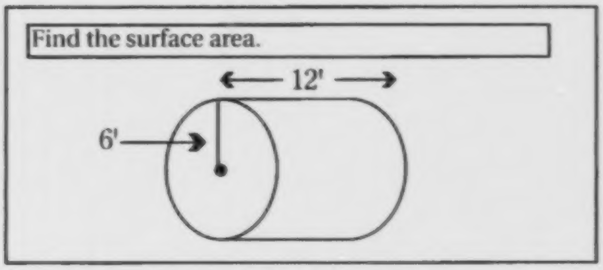


Fig. 40.6. Another scored-discussion problem

1. What did you feel or think about the scored discussion?
2. What were the positive aspects about the scored discussion?
3. What were your concerns about the scored discussion?
4. Would you make any suggestions to change or fix the process?

Journal Reflection Results

Positive aspects: About one-fourth of the students expressed the opinion that the scored discussion fostered respect and cooperation and included everyone. One-third of the students reported that they better understood the mathematics involved by either doing or watching the scored discussion. One shy girl shared, "It helped me know how to understand the meaning of the problem and how to get the answer. It also helped us stay on track to know that everyone was looking at us." Other positive aspects included learning other ways of doing a problem and seeing what does not work. A third of the students commented that scored discussion was a good model for how to work better in a group. A boy reported that "[scored discussion] made me see how others work in a group and how I can change the way I work." Another boy shared, "I liked the scored discussions. I think they made me feel more a part of the group. Our group really worked well together, and I think it made us work better." One boy wrote, "They helped me to see different ways of doing problems. We listened to other ideas better, so more got done. I like it because it helps in finding easy ways to do stuff and is rewarding when you help someone." Almost one-fifth of the students wrote, "This is fun!"

Concerns: Most concerns expressed were idiosyncratic; however, some concerns were shared. One-tenth of students commented that this behavior was phony or fake, that students acted nice just to get points, and that it would not carry over when students were not being watched. In response, I explained that uncharacteristic behavior, such as finding a reason to compliment a person I do not like or asking help of others, may appear to be fake at first, but with practice eventually something changes, so I begin to value others' thoughts and methods and then I can more easily give support to everyone. At the end of two sets of scored discussions, one girl agreed, "You were right when you said it would become more real with practice. It definitely has."

One boy wrote, "This is math class, not socializing" and "I might get graded down for something not even related to math." A few students expressed similar concerns. Whether at school or work or among family and friends, I told them, skills that research identifies as important to success in a group will serve all of us well. Shyness was mentioned as a possible problem by four students. "I think it [scored discussions] helps bring quiet people into a group," was the feeling of a girl who is not shy.

Suggestions: Three students thought that no penalty should be imposed for making a mistake. However, making a mistake cost only 1 point, whereas catching a mistake or offering a method of solving the problem earned 2 or 3 points. A student could still come out ahead by suggesting a possible strategy or catching a mistake.

Three students suggested that no grading should be used. I had told students that I would decide on a grading system after we had completed a few scored discussions. Leach's (1992) method was to take the top score in a scored-discussion group and give a percent to all other members on the basis of that score. Under that system, it would be possible for some shy students to get a failing grade when they were doing their best. I am comfortable with two scoring options. One is to average two scores of scored discussions for each student to determine her or his handicap, similar to the process used in golf or bowling. Each scored discussion is graded up or down from, or the same as, a standard 20 points, the same number of points as a quiz grade, according to the handicap score, allowing for only 1 bonus point above 20. For example, if Myra Math Whiz got scores of 16 and 18 on two scored discussions, her average is 17. The next time I observe her in a scored discussion, she gets 16 points. Since her average, or handicap, is 17, I take 1 point (17 – 16 = 1) away from the 20-point possibility, for a scored-discussion grade of 19 out of 20 (20 – 1 = 19). In the next scored discussion, Myra gets 23 points. Her handicap is 6 points below this number, but a person can only get a maximum of 1 bonus point, so her score is 21 out of 20 this time. I have currently used this method for two years and find that it works very well. Another option is to use these scored discussions for training and information purposes rather than for a grade.

Scored Discussion—First Round

After each student had participated in a scored-discussion group in front of the class during the practice round, I changed the format from scored discussions observed by the whole class to small groups working on the same problems simultaneously. I wanted to obtain a baseline score for each student, which I would use subsequently in my final evaluation. Over the next few weeks, I observed one or two groups a period, allotting three or four minutes per group—a minute per student—scoring individuals on the individual rubric until I had observed and graded each student.

Scored Discussion—Second Round

Between graded rounds, students continued to work on improving their group skills by working in cooperative groups. Then for the second round, I observed each student again within the next few weeks. I used the method in which all students worked in cooperative groups, and I sat in on one or two groups for five minutes each, checking the individual rubric for each member. I loved sitting in and observing, finding it beneficial to be able to chat afterward about what we saw, heard, and experienced. The whole-class scored discussion did not lend itself as well to this private conversation. Students worked well on a variety of tasks—homework, test reviews, reworking a mathematics contest taken earlier—helped by supporting, encouraging, and including one another. About one-third of the students' comments supported these observations. One girl noted, "We worked so everyone could keep up, even if it meant waiting a second or two. Everyone in the group helped me if I needed it. I helped people. Everyone understood the problem better." A boy wrote that "Our group worked on tasks well. We complimented each other, and we helped one another when someone had a problem."

Final Assessments

I recreated the check marks and scores from the first and second scored discussions for each student side by side and gave them these observations within a week of ending the second scored-discussion set. Each class looked these over, returned them to me, and then filled out the self-assessment form I gave them (fig. 40.7). The form had the same fourteen behaviors listed again, with no points assigned at all. Students assigned a plus sign for a positive change, a minus sign for a negative change, and a zero for no change for each behavior. I then asked them to tally the results of the negatives and positives.

With one exception, all students reported an overall increase in the positive behaviors and reduction of negative behaviors. Of all ninety-four students, the positive-behavior-change mean was +6.4, with the median being +7 out of 14 maximum positive points. Only two students had a negative score (–1). Four students had a zero score, only one having no changes and the other three changing, but the positive and negative changes were equal. These results indicated a significant positive change in students' behaviors, which justified continuing scored discussions.

The students also answered a final set of journal questions:

1. How did the scored discussion in front of class and in small groups change/affect—
    a) you?
    b) other group members?
    c) how your group worked on tasks?
2. If you have any comments or feelings about scored-discussion experiences/affects that you would like to share, please do.

Their journal entries reflected how they had changed. Many students felt more confident about their abilities and about working and sharing in a group. One girl wrote, "Well, I did learn that you could be nice and also helpful. I think that this has changed a little bit of some people's outlooks of shyness, because now I am more confident." Students also noted that they were more active and were trying to make eye contact with each other and make space for others to share. One quiet boy wrote that "I realized that you have to stop everyone and make eye contact and tell them something or explain something." Ten students commented that they worked more at supporting and complimenting fellow members. Three students thought that they tried harder in front of class in scored discussion than in small-group scored discussion.

Over half the students commented on positive changes in fellow group members. These changes included an increase in support and encouragement, in effort and confidence, and in understanding problems. Most students thought that their groups were

Self-Assessment

Name ______________________________

Final Result __________________________

Please put a plus (+) for a positive change, a minus (–) for a negative change, and a zero (0) for no change after each behavior. Tally all the positive changes and then all the negative changes. Put the final result on the line under your name. Example: 8 pluses and 3 minuses and 3 zeros equal +5.

Behaviors in a group/partnership

1. Determining a possible strategy to use ________________
2. Successfully communicating a strategy ________________
3. Correctly applying a property ________________
4. Recognizing misused properties or arithmetic errors ________________
5. Drawing another person into the discussion ________________
6. Asking a clarifying question ________________
7. Moving the discussion along ________________
8. Encouraging or praising a partner ________________
9. Making sure everyone understands ________________

1. Not paying attention or distracting others ________________
2. Interrupting ________________
3. Making an incorrect application/assumption ________________
4. Monopolizing ________________
5. Making a personal attack ________________

Fig. 40.7. Final self-assessment form

more efficient and work was well done, with more participation by all members in questioning and understanding. In general, many students found this experience to be fun and "a great way to understand better." One student reported, "It helped me realize how to understand and make others more comfortable."

Scored Discussion Benefits

From this experience of using scored discussions in my classes, I see numerous benefits.

Training: Scored discussions would be a great way to begin the school year to model and practice behaviors. The whole-class scored discussions give the audience a chance to reflect on what does and does not work while observing the scored discussion.

Assessment: Scored discussions offer students a chance to reflect on their own progress and that of their group. They also give the teacher information to help plan where to go next by observing students' understanding. Grading itself could be approached in a number of ways, from no grading, to giving a participation grade, to Leach's (1992) method of 100 percent being the top score and relating others to that score, to a handicapping approach.

Fun: Both boys and girls alike said, "I liked it a lot"; "It was fun doing the problem and helping others"; "Let's keep doing this, it's a fun way to learn."

Carryover: As I was describing this project to the literature colleague on my team, she said, "You know, I've noticed students in our small-group discussions saying more positive comments like 'I liked what you had to say' or 'Good insight.' I wonder if there's a connection or if it's just coincidence." I know what my guess is.

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Word Origins: Building Communication Connections

by Rheta N. Rubenstein

Listen to some remarks that I have heard in mathematics classrooms.

Do they sound familiar?

- Is the diameter the short one or the long one?
- I forget. I think obtuse is the wide kind of angle.
- Eight is a multiple of 24, I think, . . . or is it a factor?

AT THE middle school level, students encounter a lot of new mathematics vocabulary, some of it confusing. Sometimes I wonder whether we overwhelm students with terminology! Yet we want them to speak the language of mathematics, using standard terms that others recognize and understand. Moreover, we want the vocabulary to make sense to them.

One strategy that supports students' language learning is examining word origins. Once we as teachers are familiar with the roots of mathematical terms, we can share the etymologies as we introduce new vocabulary and help students use the roots as bridges to meanings. One bonus is that the mathematical terms are often linked with common English words, including words in other disciplines. The links promote students' general vocabulary development, as well as mathematical fluency. For example, in studying geometry, students often confuse *diameter* with *radius*. Consider the roots: *radius* comes from the same root as *ray*. Just as a ray of sunshine starts at the sun and goes out from that point, so too does the *radius* of a circle start at the center and *radiate* from there. Even *radio* is related because broadcast waves *radiate* from a tower to our receivers. *Diameter*, in contrast, comes from roots meaning *measure* and *through*. The *diameter* of a circle is a *measure* through its center. Another term, *diagonal*, shares the same prefix for *through*, as well as *gon*, meaning *angle*. These roots remind students that a diagonal is a segment that goes through an angle or corner of a polygon. Note that *polygon* itself has the same root; it means, literally, many *angles*. Table 41.1 (pp. 494–97) gives more examples for some common terms in the middle school curriculum.

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Although word origins can often be found in good dictionaries, one especially helpful resource is Steven Schwartzman's book *The Words of Mathematics* (1994), which I highly recommend. In it, Schwartzman provides etymologies of hundreds of mathematical terms, along with examples of related roots in common language. Much of table 41.1 is based on Schwartzman's information. I suggest keeping a copy of *The Words of Mathematics*, as well as your favorite dictionary, handy in class.

Students need some help getting started with etymologies. Usually, middle schoolers are just beginning to realize that language has a history and that knowing word roots can help them build vocabulary. You may want to work with an English teacher to help the students build a foundation for language studies. One way to begin is simply to share an etymology when a new concept is first introduced. It is usually easiest to tell students the meanings of the roots and help them see those roots in other words but not to dwell on the foreign-language roots themselves. Unfortunately, many roots are Indo-European, which is no longer a living language.

TABLE 41.1
Roots, Meanings, Related Word, and Notes for Some Middle School Mathematics Vocabulary

TERM, ROOT, LINK TO MEANING	RELATED WORDS	TEACHING NOTES
NUMBER		
denominator *nom:* name The denominator of a fraction tells the name or kind of fraction; e.g., *fifths* for five equal parts.	*nominate:* to name someone for an office *denomination:* the name of a religious sect, a coin or bill, a playing card, and so on	Teach denominator before numerator. First, we must know a fraction's name, then we learn how many of the parts of the fraction are of interest. Students who know French can recognize *nom* for *name*.
numerator *num:* number The numerator of a fraction tells the number of parts of interest; e.g., in 3/5 we are interested in three of those parts named *fifths*.	*numeral:* a symbol for a number *enumerate:* to count, as is done in a national census *numerous:* many; in large number	The numerator is the "numberer" of a fraction, whereas the denominator is the "namer." In measurement, the same pattern applies: We name the number before the name, or *denomination*, of the unit, as in five miles, ten square inches, or eight cubic feet.
reciprocal *re:* back, as in backward *pro:* forward In writing the reciprocal of a fraction, the numerator and denominator "go back and forth."	*reciprocal trade:* two countries buying and selling from one another; products moving back and forth *reciprocity:* agreements to honor one another's credentials	When you multiply two reciprocals, you get the number 1. You have "gone back and forth" and returned to the identity for multiplication.
percent *per:* for each *cent:* 100 Percent means *for each 100* or *divided by 100* or simply *hundredths*.	*century:* 100 years *cent:* 100th of a dollar *centennial:* 100th anniversary	*Per*, or *for each*, is a helpful translation in other contexts that signal division, such as miles per gallon, cents per gram, or feet per second.
factor *factor:* maker, doer, performer Two factors when multiplied make a product.	*factory:* a place where products are made *fact:* something that has been done *manufacture:* to make by hand (French *main*) from raw materials	Students often confuse factors and multiples because they are both parts of multiplication. It helps to point out that just as factories produce products, so too do factors produce products.
multiple *multi:* many *pli:* fold Fold a piece of paper in half, in half again, and in half again. The resulting number of pieces is eight times, or eightfold, the number of original pieces.	*multicultural:* representing many cultures *plié:* in ballet, a bending or folding of the knees *pliable:* flexible or bendable	Again, to clarify the confusion between factors and multiples, have students remember, "Every number has a multitude of multiples."

GEOMETRY		
perimeter *per:* around *meter:* measure The perimeter of a figure is the measure around it.	*periphery:* the edge or boundary of something *diameter:* measure of a circle through the center	The Greek word for *perimeter* begins with the letter *p*, which for any circle represents the quotient of its perimeter divided by its diameter.
circumference *circum:* around or bend around *ferre:* to bring, to carry The circumference of a circle is the measure you need to get around the entire circle.	*circumnavigate:* to sail around the globe *circumspect:* to look around; to be suspicious *circumlocution:* to talk around a subject, not directly about it.	A circle drawn around a polygon and through its vertices is called a *circumscribed circle.* *Scribe* means *to write or draw,* so *circumscribed* means drawn around.
polygon *poly:* many *gon:* angle or knee A polygon is a closed figure with many angles.	*polynomial:* an algebraic expression with many names or terms *knee:* a joint that bends, or makes an angle	In the Old English from which *gon* is derived, the *k* in *knee* was pronounced so it sounded more like *g* in *gon.*
polyhedra *poly:* many *hedra:* bases or seats Polyhedra are three-dimensional shapes that have many bases or seats, also called faces.	*sanhedrin:* a historical judicial body in which many representatives are "seated" *residence:* a place where one stays or is seated	The root for hedra is *sedra,* from which we get *sit, seat,* and *residence.* We call the polyhedron's seats its *faces,* but *seat* or *base* is more literal.
acute *acute:* sharp or pointed An acute angle measures between 0 and 90 degrees, which is fairly "sharp."	*acute vision:* sharp vision *acupuncture:* procedure using sharp needles *acrobat:* one who performs near the sharp or pointed part of a tent	We can demonstrate acute pain by bending sharply to make an acute angle with our bodies.
obtuse *ob:* against or toward *tuse:* beat When you beat a knife against a surface, it becomes dull or blunt. An obtuse angle measures between 90 and 180 degrees, which is fairly "dull."	*obscure:* literally, *to conceal against;* something that is lacking light; hidden; not easy to understand	Another word of interest, but possibly one you would not mention in class, is *obscene,* which means *off stage.* In Greek theater, actions that were obscene were not seen!
ALGEBRA		
variable *wer:* a raised spot on the skin; a change from normal A variable in algebra is a symbol that can take on various, or changing, quantities.	*variable weather:* conditions that are changing *variation* (in statistics): how values are dispersed, or changing, from one extreme to the other	Many educators suggest giving students early experiences with variables that vary (Pegg and Redden [1990]; Demana and Leitzel [1988]. Avoid limiting early experiences to "finding an unknown." ➤

algorithm Ben Musa al-Khowarizmi was a ninth-century mathematician who wrote books about what we refer to today as *arithmetic* and *algebra*. The end of his name, al-Khowarizmi, meaning *from the town of Khowarizm*, is the source of the word *algorithm* for a step-by-step procedure, which he often showed in his works.	*algebra:* One of al-Khowarizmi's books began with the Arabic word *al-jabr*, meaning *the reunion of broken parts*. In solving equations, we are working from parts or pieces of information to wholes, or finding the missing pieces.	Arabic is also the source of our word *zero*. It is from the word *sifr*, for *empty*. The Arabs got the word from the Hindus, who first conceived of a place-value numeration system with a symbol to stand in a place with no value.
coordinate *co:* together with *ordinate:* a straight row When you put things in order, they are in a straight row. Coordinates are an ordered set of numbers used to locate a point.	*cooperate:* to operate or work together *coordinates:* clothes that go together	Longitude and latitude from another system of coordinates that tell a location. In other words, where in the world are we?
axis *axis:* axle or pivot An axis is a line around which a coordinate system "pivots."	*axle:* a pin or rod on which a wheel revolves	Just as a car wheel's turning requires an axis, so too does locating points in a coordinate first require axes.
origin *origin:* the beginning; the source	*Orient:* where the sun rises or originates; refers to the Far East	In China, the name for Japan, *Yat Bun*, literally means *day root*, or the origin of the day. Notice how the Chinese look east to the sunrise in Japan, just as Europeans look east to the sunrise in China. Similarly, *Europe* comes from a Semitic word meaning *(land of the) setting sun*.
DATA		
statistics *status:* a condition or standing *Statistics* means the collection, organization, analysis, and interpretation of quantitative data.	*status:* condition or standing of a person or thing *stature:* natural height when a person is standing	When you are done with statistics, you know how things "stand."
graph *graphein:* to write, to scratch A graph is a diagram or written picture of quantitative information.	*telegraph:* writing from a long distance; literally, *far writing* *autograph:* writing one's own name; literally, *self-writing* *graphology:* study of handwriting *crab:* an animal that scratches the ground as it walks	*Graph* and its cousin, *gram*, are pervasive in mathematics and science. Challenge students to find other examples, such as *diagram, histogram, scattergram, pictograph, bar graph, electrocardiogram, encephalogram, spirograph, computer graphics*, and so on.

average *awar:* fault or blemish From the Arabs, who were extensive traders, *average* originally meant *goods damaged in shipping*. Later, the meaning became *the financial loss due to damaged goods* and, eventually, *the the portion of the loss borne by each investor*. Now *average* means an *equal portion*.	*damage:* loss resulting from injury, as well as the legal compensation for such a loss (Damage was originally a synonym of *average*.)	Students can be helped to visualize *average* using materials. For example, show students stacks of seven blocks, two blocks, and three blocks. Ask them how to make equally tall stacks. Help them see that the average is the height needed for each stack to have an equal number of blocks.

Invite students to brainstorm to identify *cognates*, or words that appear to be related, then check dictionaries to verify their suggestions. Students for whom English is a second language may share related words in other languages, too. Once students get the idea, they can be invited to dig deeper using dictionaries and the Internet. Etymologies can be part of study-skills activities (UCSMP 1995), can serve as sources for student journal entries (Thiry 1990), can be used as components of student projects, or can be included in student-developed bulletin boards or cartoons. Thompson and Rubenstein (2000) provide more specific ideas for using etymologies.

Incorporating word origins into the curriculum produces multiple benefits. The main advantage is that mathematical terms often mean precisely what they say. When we recognize the Latin, Greek, Indo-European, Arabic, or other language roots from which words stem, their meanings are clearer. Another benefit is that students who are verbal learners, as in the multiple intelligences of Gardner (1983), often find verbal connections particularly helpful in gaining access to the mathematics. Etymological links help English-as-a-second-language students learn common and technical vocabulary in coherent ways. Parents, looking ahead to college-entrance examinations, are usually pleased to know that language development is part of the curriculum. Finally, the study of word origins is a wonderful opportunity for teachers to collaborate across disciplines. Along with reading and writing across the curriculum, why not integrate language across the curriculum? Word origins appear in all disciplines. Teachers who are aware of terms that students are learning in other classes can use etymologies to support student learning in other areas.

Most of all, word origins are fun! Learning how seemingly unrelated words in our lives are connected is often surprising and exciting. For example, did you ever realize that *angle* and *ankle* sound alike? This similarity is not surprising when one discovers that they both stem from a root meaning *a small bend*; in other words, where the foot meets the leg, an ankle makes an angle! To find more interesting and enlightening connections, begin exploring word origins, sharing them with students, and enlisting students as fellow explorers.

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Reasoning and Proof

INSTRUCTIONAL programs from prekindergarten through grade 12 should enable students to —

- *recognize reasoning and proof as fundamental aspects of mathematics;*
- *make and investigate mathematical conjectures;*
- *develop and evaluate mathematical arguments and proofs;*
- *select and use various types of reasoning and methods of proof.*

"Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and . . . at all grade levels, students should see and expect that mathematics makes sense" (NCTM, 2000, p. 56). The following articles illustrate how appropriate tasks and carefully orchestrated classroom discourse set the stage for students' engagement with the processes of reasoning and proof.

Maylone describes how counterintuitive problems can be used to spark discussion and debate among middle school students. Because solutions to these problems fly in the face of common sense, students must rely on understanding the mathematics of each problem to verify the correctness of the results. Wrestling with the resolution of such problems helps students understand the value of mathematical argument.

Peressini and Knuth discuss the importance of constructing appropriately rich tasks to elicit students' use of mathematical justification. The authors describe a particular task of establishing racing routes across a square playground for five separate participants, then present two plausible solutions to this situation. Students are requested to comment on each solution and then construct and defend an alternative race scenario. Examples of students' responses to this task illustrate how such work can be used to help strengthen students' understanding of what counts as acceptable mathematical explanation and justification.

Using Counterintuitive Problems to Promote Student Discussion

by Nelson J. Maylone

THE NCTM's *Curriculum and Evaluation Standards* document (1989) has sparked a revolution in curriculum and textbook design and, to some extent, the evaluation of students. The Standards encourage problem solving in context; increasing opportunities for students to discuss mathematics; and actively involving students in exploring, conjecturing, analyzing, and applying mathematics. Recommended instructional techniques include promoting group work, making presentations, conducting experiments, and working with manipulatives.

Using counterintuitive mathematics problems helps to keep adolescent students actively involved in their mathematics education. Counterintuitive problems can be defined as answers and solutions that may not, at first, seem right to students or adults. A classic example is the Birthday Match Problem:

What is the minimum number of people, randomly chosen, who need to be assembled in a room so that the chances are at least 50 percent that two of them have exactly the same birthday? (Month and day, not year)

Typical guesses include 365, which is the number of days in most years, and 183, which is about half of 365. The correct answer is surprising, but more on this problem is found at the end of this article.

Counterintuitive problems are not necessarily trick questions; rather, before discussion, their answers appear to fly in the face of common sense. Therein lies one of their chief benefits: these problems promote momentary confusion; then debate; and, finally, understanding. An identifying characteristic of counterintuitive problems is that even after a solution has been verified, one may have difficulty ignoring that little voice that keeps whispering, "But that answer can't be right!" That dissonance is part of the fun of using counterintuitive problems, since adolescents tend to be passionate about defending that which they see as right. Even reticent students will find themselves being drawn in by these challenges. Motivated students not only will pay attention but will think and reflect, be eager to contribute, and perhaps most important, continue to want to contribute even after being proved "wrong."

Any middle school teacher can confirm that adolescents enjoy debating. Discussion and debate engage students and promote new thinking, reinforce learning, and offer important social erudition opportunities. Verbal interaction can help teach students how to listen, be patient, respond appropriately to others' comments and opinions, and be concise and persuasive. Using counterintuitive problems can also provide

> Attorneys Barr and Writ are two highly competitive injury lawsuit lawyers. They are so competitive that twice a year, at the end of June and December, they compare their trial won-lost records, with the "winner" treating the other lawyer to dinner.
>
> For the first half of a recent year, Barr's winning percent, the number of cases won divided by the total number of cases handled, was better than Writ's winning percent. For the second half of that year, Barr again had a better winning percent than Writ.
>
> Is it possible that attorney Writ had a better winning percent than attorney Barr for that entire year? If so, how? If not, why not?

Fig. 42.1. The Busy Lawyers Problem

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opportunities to incorporate students' intuitive solutions, thereby increasing learning. Furthermore, if students are enthusiastically engaged, discipline problems will be minimized.

Several problems follow. Some are variations of ancient classics; some are original—all are counterintuitive. The middle-grades-mathematics teacher will want to share them with students and to allow the subsequent debate to rage in the classroom.

The Busy Lawyers

The classic Busy Lawyers Problem is shown in figure 42. 1. Please read it and answer the question at its end before proceeding.

The answer is yes, Writ could have had a better all-year winning percent than Barr in spite of Barr's having the better first half and the better second half.

Intuition loudly screams that Writ could not possibly have had a better yearlong winning percent than Barr. When first confronted with this problem, one seventh grader asserted that "Barr beat Writ twice. There is no way that two wins could be a 'loss.'" Barr beat Writ in the first half, Barr beat Writ in the second half; case closed.

Although those ideas are good opening arguments from students, the teacher should challenge them to push their thinking beyond the obvious. If students do not bring up the point on their own, the teacher should ask them whether the number of cases that each lawyer handled is relevant. (It is.) This question may spark some students to examine the situation more closely. Here are other questions that the teacher may want to pose if students are struggling with this issue:

1. Barr's yearlong winning percent will fall between what two other percents? (Answer: Barr's first-half and Barr's second-half winning percents.)
2. Using the same logic, where will Writ's yearlong winning percent lie?
3. Will Barr's and Writ's yearlong winning percents always be exactly halfway between their half-year winning percents? If not, what factor might determine to which half-year percent each lawyer's yearlong percent is closest?

The teacher's purpose here is to encourage students to rely on their intuition, which says no, each lawyer's yearlong winning percent may not lie exactly halfway between their half-year percents. This revelation may open up the possibility that Barr may have the better yearlong winning percent.

Table 42.1 describes a possible scenario in which Barr wins the individual halves of the year but Writ, nonetheless, becomes the yearlong champion. Note that the winning percent is given in baseball-statistics style, that is, as a decimal shown to the thousandths place.

TABLE 42.1
One Possible Solution Scenario

	LAWYER	WINS FROM CASES TRIED	WINNING PERCENT
First half of the year	Barr	60 out of 100	0.600
	Writ	29 out of 50	0.580
Second half of the year	Barr	40 out of 50	0.800
	Writ	79 out of 100	0.790
Entire year	Barr	100 out of 150	0.667
	Writ	108 out of 150	0.720

An everyday example may help to clarify the issue for students:

> Terry is given only one mathematics test during the first semester, and he earns a score of 100 percent. During the second semester, twelve tests are administered, and Terry consistently earns a score of 80 percent on each one. Assuming that the teacher assigns equal weight to all thirteen tests, is it fair to take the first-semester score of 100 percent and average it with the consistent second-semester score of 80 percent for a yearly average of 90 percent?

Terry might wish it so, but that approach is bad mathematics.

Students should walk away from this activity with the understanding that, generally, one should not average averages. As one student observed, "I think denominators matter."

Why Isn't Pat Broke?

Discussing the problem shown in figure 42.2 will help clarify the truth that 10 percent of a dwindling amount is an ever-dwindling amount. Ten percent of something remains a consistent size only if the original amount under consideration does not vary. The teacher may want to ask students this series of questions:

1. "Reduce $20 by 50 percent. What do we have?" Most students will be able to give the correct response of $10.
2. "Now reduce that $10 by 50 percent." A moment of hesitation is likely as students see that this new reduction takes them to $5.
3. "We have done two reductions of 50 percent. Shouldn't two reductions of 50 percent take us to $0?" Most students will be able to state that, no, we should not now be at zero; we took "50 percent off" two different amounts. The hope is that students will be able to recognize that the same phenomenon is at work as Pat's company loses money.

> The net worth of Pat's Candy Company at the start of a particular year was $32,000.00. Because of declining sales, Pat's company began losing 10 percent of its worth each month. At the end of the year, the company was still worth $9,037.74. How can that be? Wouldn't losing 10 percent per month for just *ten* consecutive months, let alone twelve, leave Pat with a worth of $0?

Fig. 42.2 The "Why Isn't Pat Broke?" Problem

The teacher's goal is to lead students to state in their own words the phenomenon noted through the questions. One sixth-grade student commented, "When you chop off 10 percent of what you have, you have something smaller, and 10 percent of it will be smaller, too." That observation is praiseworthy. It indicates that the student has understood the essence of this mathematical phenomenon.

The Raffle

Consider the problem shown in figure 42.3. Answer the question before proceeding. The answer is that we cannot say who will win for sure, but it probably will not be Jesse! Students will object to this statement. In one sixth-grade classroom, one student stated, "I know that Jesse might not win, but Jesse has lots more tickets than anyone else, so that person will win probably." Although it is correct that Jesse is 400 times as likely to win as anyone else, it is still true that someone else will probably win. Here is why.

> To raise money for a new outdoor school sign, the principal decided to hold a raffle. Exactly 1000 tickets would be sold for $1 each. One winning ticket would be chosen, the prize being free school lunches for a year. The first 600 tickets were sold to 600 separate individuals, and the remaining 400 were all sold to Jesse Bernhoot.
>
> Question: Who will likely win the raffle?

Fig. 42.3 The Raffle Problem

The sold tickets can be seen as forming two blocks, one of 400, another of 600 tickets. When those 1000 tickets are thoroughly mixed together in a drum, neither the drum nor the ticket drawers know that Jesse has purchased so many. It is more likely, although by no means certain, that the winning ticket will come from the block of 600 than from Jesse's block of 400.

This problem is counterintuitive because of the following true, but apparently contradictory, statements:

1. Jesse is 400 times as likely as anyone else to win.
2. Someone other than Jesse is still more likely to win.

Why? Because Jesse is indeed 400 times more likely than any other individual ticket holder to win the raffle, but Jesse is not 400 times more likely than all the other 600 ticket holders to win.

A good classroom followup to this problem would be to discuss the mathematical wisdom of playing regional and state lotteries. Perhaps students have read about people who have spent their life savings buying thousands of tickets for one day's drawing, their thinking being that if they buy 5,000 tickets, and no one else has bought more than, say, 20 for that drawing, they are sure to win. After all, 5,000 is many times more than 20. Like Jesse, however, that individual will probably not win. Although he or she may have purchased 5,000 tickets, far more than anyone else, total sales might equal 1,000,000 tickets, nearly assuring that someone else will win.

Some lottery advertisements use a slogan, such as, "Someone's gonna win!" Truth in advertising would require the follow-up line "And it probably won't be you!" As a popular bumper sticker points out, "The lottery is a tax on the mathematically challenged."

Rolling the Block

The interesting problem shown in figure 42.4 is for students who know that the circumference of a circle is pi times the diameter. They are likely to propose an answer of 31.4 inches, the circumference of one log. Students should be given credit for knowing and applying the circumference formula and for demonstrating an understanding of the meaning of pi. They should also be tactfully informed that they are incorrect. An explanation follows.

Workers of an ancient civilization attempted to roll a stone block along a series of logs. The figure below illustrates the situation. The diameter of each log is 10 inches. If the block is pushed forward, how far in relation to the ground will the block travel if each of the underlying logs makes one complete roll? (Use 3.14 for pi.)

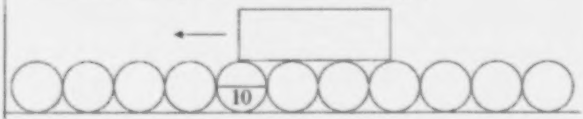

Fig. 42.4 The Rolling the Block Problem

Rolling over the tops of 10-inch-diameter logs as they make one turn will move the block 31.4 inches. This situation fails to take into account that the logs themselves also move forward by 31.4 inches. The forward progress of the block is combined with the forward motion of the logs themselves, so the block will roll 31.4 + 31.4 inches, or 62.8 inches, for each log.

The Careless Cat

Many students will want to make a sketch of the problem in figure 42.5. Encourage this tactic, as it is an appropriate problem-solving strategy. Students' sketches may look something like figure 42.5. The zigzags represent the cat's upward, then downward, progress and regression.

At precisely noon, a careless cat falls into a six-foot-deep hole, which has very slippery sides. Immediately, the cat tries to climb out and makes progress at this rate: it climbs up two feet per minute, then slips back one foot over the next minute. What time will the cat emerge from the hole?

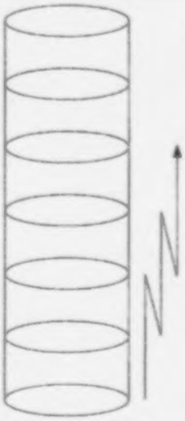

Fig. 42.5. The Careless Cat Problem

One eighth grader who was working on this problem stated, "Going up two feet in a minute, then going down one foot the next is like going up one foot every two minutes." At that rate, the cat would emerge from the six-foot hole at 12:12 p.m.

However, once the cat reaches the top of the hole at 12:09 it no longer falls back. Continuing with the zigzag pattern suggested in figure 5 may clarify this problem for skeptical students.

Birthday Match

Here are some comments on the Birthday Match Problem mentioned previously. The teacher may want to backtrack for a moment and ask, "How many people would need to be in a room for it to be a certainty that two of them have the exact same birthday in a nonleap year?" Many students will quickly give the correct answer of 366. The teacher can then present the Birthday Match Problem:

> What is the minimum number of people, randomly chosen, who need to be assembled in a room so that the chances are at least 50 percent that two of them have exactly the same birthday?

Students may make a first guess of 183—half of 366. The teacher can expect other wild guesses in excess of the correct answer of 23.

Even students who have developed a sense of caution when approaching such questions as these may state that the number 23 simply cannot be right. "Twenty-three is hardly any people compared with 365" was how one eighth grader put it. One way to convince them is to have them check several other classrooms in the school for common birthdays.

Even after empirical verification of the answer, students will not be clear as to why 23 is correct. Here is a brief explanation, offered with the assumption that students have at least some basic prior knowledge of probability.

What are the chances that a birthday match will not occur if one person stands alone in a room? The chances are 100 percent—no one is in the room to match up with! That probability can also be shown as

365/365; 365 possibilities divided by 365 matches that are sure not to happen.

What is the no-match probability if two people are in the room? Since 364 nonmatching possibilities exist, we have chances of 365/365 × 364/365, multiplying the individual probabilities together, giving about .997, or near certainty.

With three people in the room, the no-match chances become 365/365 × 364/365 × 363/365, or about .989, which is still near certainty, but not as near. Continuing in this way, we would at some point compute a no-birthday-match probability of approximately .5, or 50 percent, meaning that the chances of having a match are also about 50 percent. In fact, we would need to go "twenty-three fractions deep" for the probability to hit approximately 50 percent. The computation can easily be verified with a calculator.

Summary

Use of counterintuitive problems such as those presented here can promote meaningful thought in middle-grades classrooms and can provide a framework for substantive, intelligent discussion. The broader theme of course is student engagement, in keeping with the NCTM's Standards.

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The Role of Tasks in Developing Communities of Mathematical Inquiry

by Dominic Peressini
and Eric Knuth

As the third millennium approaches, efforts to improve mathematics instruction and assessment continue. We have made progress in enriching students' mathematical experiences, and the lessons we have learned from the past are reflected in the latest NCTM *Standards* (1998). These updated standards build on the three previous NCTM *Standards* documents (1989, 1991, 1995) and research findings from the last three decades; they contain reform recommendations to guide mathematics teachers into the new millennium. One central idea that continues to steer our reform efforts is a vision of school mathematical practice that reflects the essence of practice in the discipline of mathematics. All children should have ample opportunities in their classrooms to work as young mathematicians. In this article, we examine the nature of mathematically rich tasks and varied ways in which students respond to these tasks. We then examine approaches for using such tasks to foster classroom inquiry that engages children in mathematical practice.

The Activity of Doing Mathematics

We have heard many accounts of how reform-based mathematics classrooms differ from traditional classrooms. This juxtaposition has become "old news" in the mathematics-education literature, and our focus is now directed toward enacting these visions of reform. The *Standards* calls for teachers to help all students learn to value mathematics, become confident in their abilities to do mathematics, become mathematical problem solvers, learn to reason and communicate mathematically, and establish classrooms where teachers and students engage in rich discourse about important mathematical ideas and explore interesting problems grounded in meaningful contexts (NCTM 1998).

In the midst of this vision, it is sometimes difficult to discern what it means for students to engage in mathematical activity. We turn to the work of Imre Lakatos (1976) to clarify what we mean by students' engaging in activity that reflects the practice of mathematics. Lakatos argues that doing mathematics is ultimately a human activity in which "courage and modesty" play central roles (Lakatos 1976, p. 30). These human virtues come into play as mathematicians make conjectures about relationships, then follow a "zigzag" path as they prove and disprove their conjectures and the premises and assumptions related to these conjectures. Lampert (1990, p. 31) summarizes Lakatos's perspective on mathematical activity:

> From the standpoint of the person doing mathematics, making a conjecture (or what Lakatos calls a "conscious guess") is taking a risk; it requires the admission that one's assumptions are open to revision, that one's insights may have been limited, that one's conclusions may have been inappropriate. Although possibly garnering recognition for inventiveness, letting other interested persons in on one's conjectures increases personal vulnerability. Courage and modesty are appropriate to participation in mathematical activity because truth remains tentative, even as the proof of a conjecture evolves.

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Embracing this perspective, we focus on how students invoke their courage and modesty to make conjectures about mathematical relationships; how they justify these conjectures; how these justifications evolve by following a zigzag path; and most important, how this activity unfolds and becomes public discourse in the social context of the classroom. We use the term "community of mathematical inquiry" to refer to classrooms in which these activities are valued by both students and teachers. Schoenfeld (1992, p. 345) elaborates on the importance of developing these types of classrooms:

> If one hopes for students to achieve the goals specified here—in particular, to develop the appropriate mathematical habits and dispositions of interpretation and sense-making as well as the appropriately mathematical modes of thought—then the communities of practice must reflect and support those ways of thinking. That is, classrooms must be communities in which mathematical sense-making, of the kind we hope to have students develop, is practiced.

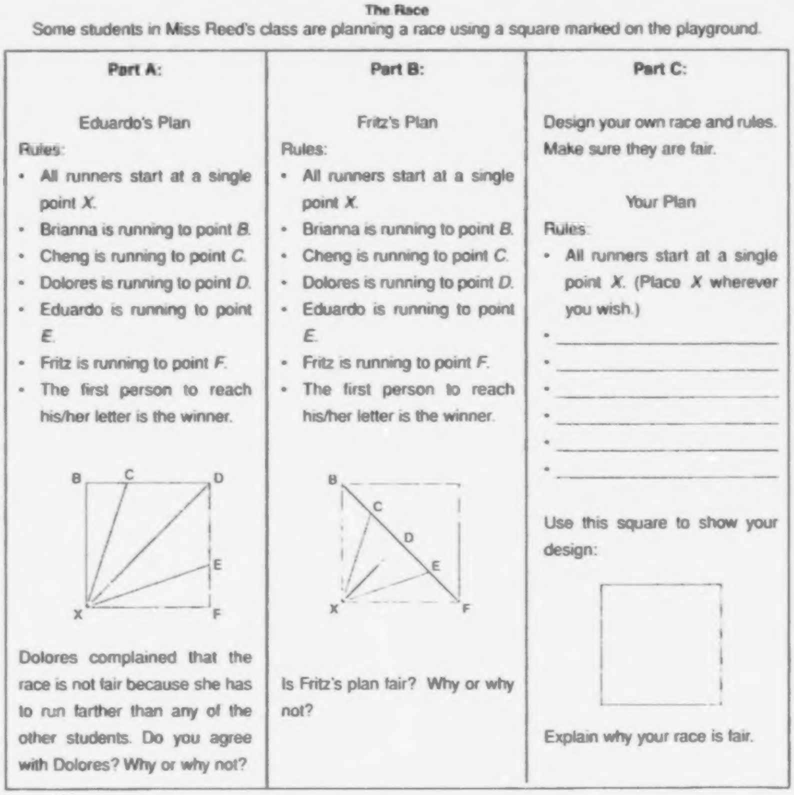
We have identified three processes in which teachers engage as they nurture communities of mathematical inquiry: (1) posing rich mathematical tasks, (2) fostering student discourse about these tasks and their solutions, and (3) reflecting on the tasks and discourse to maximize mathematical activity and student understanding.

A Mathematically Rich Task

The Race (see fig. 43.1) is a mathematically rich task that teachers have created and used to assess their students' mathematical knowledge. This task was developed as part of the Wisconsin Performance Assessment Development Project (WPAD) as part of the Wisconsin Student Assessment System. The research and development were conducted at the Wisconsin Center for Education Research under the direction of Norman Webb.

The Race serves as the focus of our discussion and fits our criteria for mathematically rich tasks, that is, tasks that encourage a range of solution approaches, address significant mathematical concepts, require students to justify their explanations, and are open ended. The Race task is also set in a familiar context for elementary school children—running a race on a school playground. The task uses the children's interest in the notion of fairness to engage them and motivate them to solve the problem using important geometric and measurement concepts and procedures. Students are given two different race plans—conjectures—to consider for fairness, and they are asked to justify their conclusions. The Race is open ended because multiple ways can be used to solve the task, multiple answers are possible, and the final requirement asks students to design their own race plans, that is, make their own conjectures, and justify whether their plans are fair.

Classroom teachers of grades K–12 can find mathematically rich tasks similar to The Race in a variety of places. Collections of mathematics tasks and ideas for creating tasks can be found in such publications as the NCTM's Addenda Series and from *Great Tasks and More* from the National Council of Supervisors of



The Race

Some students in Miss Reed's class are planning a race using a square marked on the playground.

Part A:

Eduardo's Plan

Rules:

- All runners start at a single point *X*.
- Brianna is running to point *B*.
- Cheng is running to point *C*.
- Dolores is running to point *D*.
- Eduardo is running to point *E*.
- Fritz is running to point *F*.
- The first person to reach his/her letter is the winner.

Dolores complained that the race is not fair because she has to run farther than any of the other students. Do you agree with Dolores? Why or why not?

Part B:

Fritz's Plan

Rules:

- All runners start at a single point *X*.
- Brianna is running to point *B*.
- Cheng is running to point *C*.
- Dolores is running to point *D*.
- Eduardo is running to point *E*.
- Fritz is running to point *F*.
- The first person to reach his/her letter is the winner.

Is Fritz's plan fair? Why or why not?

Part C:

Design your own race and rules. Make sure they are fair.

Your Plan

Rules:

- All runners start at a single point *X*. (Place *X* wherever you wish.)
- ______________________
- ______________________
- ______________________
- ______________________
- ______________________
- ______________________

Use this square to show your design:

Explain why your race is fair.

Fig.43.1. The Race task

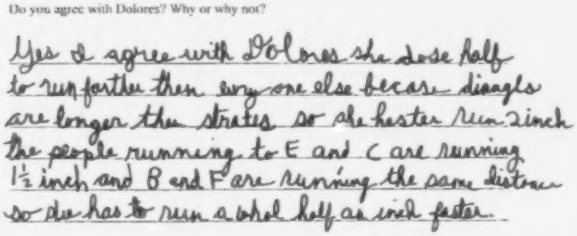
Mathematics (NCSM 1996). Web sites for mathematics tasks include www.wits.ac.za/ssproule/pow.htm and forum.swarthmore.edu/pow. In addition, possible ways to assess and evaluate these more open-ended tasks, including examples of scoring rubrics, can be found in Lambdin, Kehle, and Preston (1996), Peressini and Webb (1999), and Webb (1993). Perhaps most important, discussions of how to think about these tasks in mathematics classrooms and ways to foster mathematical reflection and discourse among students can be found in Silver, Kilpatrick, and Schlesinger (1990) and Brown and Walter (1990).

Student's Mathematical Activity

In this section, we examine excerpts of six fourth-grade students' responses to The Race task and analyze their responses in terms of the mathematical justifications provided by the students.

Jesse's response

Jesse's response to part A (see fig. 43.2) demonstrates a reliance on physical measurements as evidence for concluding that the race is not fair. Interestingly, Jesse begins his justification by stating that "diagonals are longer than straights." He is most likely referring to the "diagonals" ($\overline{XC}, \overline{XD}, \overline{XE}$) of the square and its sides ($\overline{XB}$, $\overline{XF}$). He then sees the need to support this statement with further proof—the actual measurements. Thus, Jesse provides evidence at two levels. The first level is based on his knowledge of the geometric properties of a square; the second level is empirical, suggesting that Jesse may have been motivated in his justification by the kinds of evidence that are accepted in his mathematics classroom community. Although he is correct in stating that the race in part A is unfair, Jesse seems to have confused the physical measurements of the diagram with the actual distances each runner travels (e.g., he claims that Dolores runs 2 inches).



Do you agree with Dolores? Why or why not?

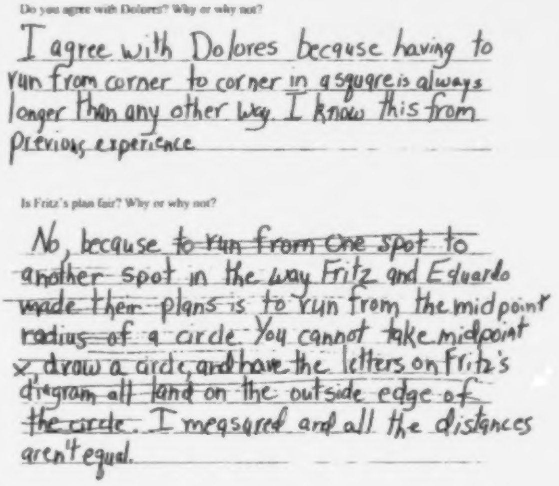
Yes I agree with Dolores she dose half to run forther then evry one else becase diagls are longer then strates so she haster run 2inch the people running to E and C are running $1\frac{1}{2}$ inch and B and F are running the same distance so she has to run a whol half an inch faster.

Fig.43.2. Jesse's response to part A

Amy's response

In concluding that the race in part A is not fair, Amy responds, like Jesse, by relying on her knowledge of the geometric properties of a square (see fig. 3). Yet unlike Jesse, Amy does not provide further mathematical justification for this claim.

In part B, Amy contends that Fritz's race plan is also unfair, yet rather than appeal to her knowledge of geometry as she did in part A, she justifies her conclusion by measuring the actual length of each segment (see fig. 43.3). Her initial response, which she subsequently crossed out, is noteworthy for the level of geometric knowledge that she demonstrates. Amy's crossed-out response suggests that she has an understanding, although a fragile one, of the fundamental property that every point on a circle is equidistant from its center. She uses this knowledge to conclude that with *X* as the center of a circle (she refers to *X* as the midpoint), it is impossible for the remaining points to lie on the circle and, as a result, the distance to each finishing point must be different.



Do you agree with Dolores? Why or why not?

I agree with Dolores becquse having to run from corner to corner in a squqre is always longer than any other way. I know this from previous experience

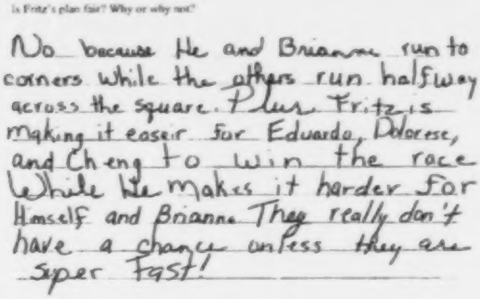
Is Fritz's plan fair? Why or why not?

No, because ~~to run from one spot to another spot in the way Fritz and Eduardo made their plans is to run from the midpoint radius of a circle. You cannot take midpoint x, draw a circle, and have the letters on Fritz's diagram all land on the outside edge of the circle.~~ I measured and all the distances aren't equal.

Fig.43.3. Amy's responses to parts A and B

Maria's response

Although Maria recognizes that the race in part B is unfair, she does not provide mathematical evidence to support her claim; rather, she relies solely on perceptual evidence (see fig. 43.4). In other words, Maria's conclusion is based on visual evidence alone: running to the corner of a square looks farther than running "half way across the square." This justification is quite different from saying that a diagonal of a square is longer than its sides. Although this fact is visually apparent, it is also based on the geometric properties of a square that many fourth-grade students recognize as being true. Interestingly, Maria introduces the concept of speed as a factor that could also determine the outcome of the race—if both Fritz and Brianna were "super fast," they might have a chance to win.



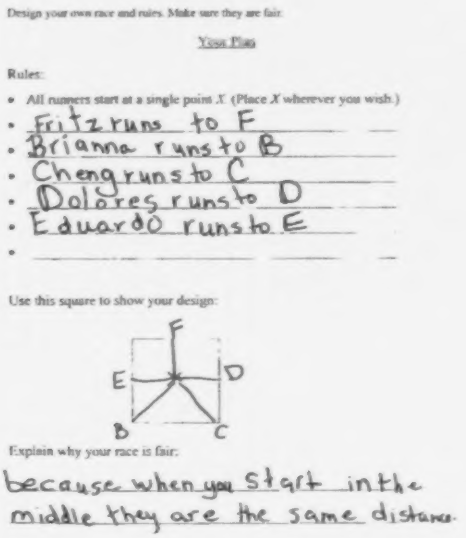
Is Fritz's plan fair? Why or why not?

No because He and Brianna run to corners while the others run halfway across the square. Plus Fritz is making it easeir for Eduardo, Dolorese, and Cheng to win the race While He makes it harder for Himself and Brianna They really don't have a chance unless they are Super Fast!

Fig. 43.4. Maria's response to part B

Cam's response

Cam's response to part C (see fig. 43.5) suggests that he may have a misconception about the geometric properties of a square. He apparently believes that a square has a middle that is equidistant from every point on the square. Although his design is fair among Eduardo, Fritz, and Dolores and is fair between Brianna and Cheng, among all five it is not fair. He has incorrectly assumed that the distance these three students run is the same as the distance that the other two students run.



Design your own race and rules. Make sure they are fair.

Your Plan

Rules:

- All runners start at a single point *X*. (Place *X* wherever you wish.)
- Fritz runs to F
- Brianna runs to B
- Cheng runs to C
- Dolores runs to D
- Eduardo runs to E
-

Use this square to show your design:

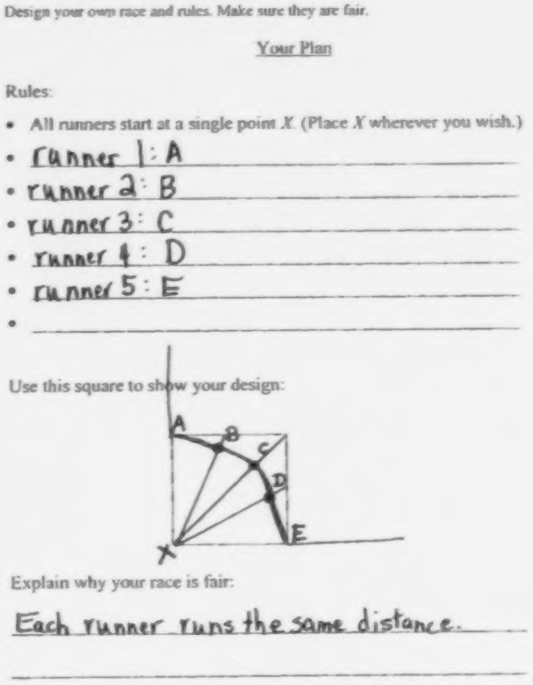
Explain why your race is fair:

because when you start in the middle they are the same distance.

Fig. 43.5. Cam's response to part C

Joshua's response

Joshua has drawn a diagram that adequately depicts a fair race, but he has not provided any evidence to show that the distances are indeed the same (see fig. 43.6). We should point out, however, that in his responses for parts A and B, he does measure the various distances in justifying his conclusions. It might be reasonable to assume that he has measured the distances in part C, as well, but has not written them down.



Design your own race and rules. Make sure they are fair.

Your Plan

Rules:

- All runners start at a single point *X*. (Place *X* wherever you wish.)
- runner 1: A
- runner 2: B
- runner 3: C
- runner 4: D
- runner 5: E
-

Use this square to show your design:

Explain why your race is fair:

Each runner runs the same distance.

Fig. 43.6. Joshua's response to part C

Using Discourse to Foster Communities of Mathematical Inquiry

These students' responses reveal a variety of information about their understanding of mathematics and, in particular, their ability to engage in mathematical activity. Indeed, this task can be used as a springboard to move closer to the types of enriching mathematical tasks discussed previously. The task gives teachers an opportunity to engage their students in rich discourse as they share their solutions. The essential role of discourse in broadening mathematical tasks and building communities of mathematical inquiry is captured in one of NCTM's earlier Standards documents (1991, p. 54):

> Because the discourse of the mathematics class reflects messages about what it means to know mathematics, what makes something true or reasonable, and what doing mathematics entails, it is central to both what students learn about mathematics as well as how they learn it.

Looking back at the different responses presented in this article, we can envision how mathematically fruitful a class discussion of these responses would be to strengthen and extend the thinking of individual students presenting their work and for the rest of the class, as well. The discussion might involve students' asking Jesse to clarify what he meant by "straights" or students' attempting to figure out how fast each runner would have to run to make the race in part B fair (see Maria's response). Other students might present solutions that are similar to Amy's crossed-out solution to part B, which would help her confirm that her initial thoughts were correct and help clarify her use of mathematical language (i.e., "midpoint" and "radius").

In addition, the discussion might address transgressions of established social and mathematical norms for the classroom. These guidelines are the "normative understandings of what counts as mathematically different, mathematically sophisticated, . . . [and] what counts as an acceptable mathematical explanation and justification" (Yackel and Cobb 1996, p. 461). For example, students might discuss differences in the two justifications that Jesse provided for part A—one based on actual measurements and the other, on a property of a square. Would the latter be perceived as more sophisticated or as requiring specialized knowledge and, thus, not valid? Students might object on mathematical grounds that Amy's justification in part A is based on her previous experience. Or perhaps students might call on Maria, Cam, and Joshua to provide evidence for their conclusions. For Cam's response, a discussion of his solution might result in the class's exploring ways to disprove his conclusion and connecting his notion of a square's middle with that of a circle—an important aspect of the mathematics underlying this task.

Although this task was originally designed for performance assessment, we can easily imagine several scenarios for presenting the task to students in ways that would foster discourse. For example, teachers might assign students part A, then facilitate a discussion of their solutions to it. In this manner, teachers not only get a sense of individual students' understanding of the underlying mathematics but can also identify students who may need additional guidance. Next, teachers might have students do part B while working in small groups, giving students the opportunity to hear other solutions and giving teachers or other students the opportunity to offer assistance. Finally, teachers could assign part C as homework or as an assessment to be discussed the following day. Teachers might also consider assigning all three parts to students as they work in small groups, but the whole-class discussion might focus solely on solutions to part C.

Another approach would be to assign part A to some groups and part B to other groups. After each group arrives at a solution, the groups that were assigned part A would evaluate the solutions from the groups that were assigned part B, and vice versa. In this scenario, students have the opportunity to "develop and evaluate mathematical arguments" (NCTM 1998, p. 80). After discussing the solutions and evaluations as a class, teachers might assign part C to all the groups to be done during the following day. The solutions to part C could then be reviewed and evaluated as a class. As a culminating activity, the results could be "published" in some way, reflecting the practice of mathematicians in which "a proof becomes a proof after the social act of 'accepting it as a proof' " (Manin 1977, p. 48).

Closing Remarks

Examining and discussing solutions and justifications for mathematically rich tasks can serve several purposes. First, classroom discussions give teachers a greater opportunity to assess their students' understanding of mathematics, compared with using written assessments alone. When students provide more detailed explanations, ask questions, or offer suggestions—that is, engage in mathematical activity—teachers see the dynamic workings of their students' minds. Second, in-class discussion supplies a venue for further exploration of mathematical concepts, which can foster students' conceptual understanding. This exploration may result in students' generating unexpected solutions and justifications, because these types of tasks are open ended and a teacher cannot possibly think of every solution strategy that students might attempt. Finally, discussion of student solutions establishes a context in which all students are engaged in legitimate mathematical activity. Such classroom interactions give students opportunities to "make conjectures and present solutions; explore examples and counterexamples to investigate a conjecture; try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers; [and] rely on mathematical evidence and argument to determine validity" (NCTM 1991, p. 45).

In summary, we find that teachers must not only teach content but also show students how to draw on their courage and modesty to participate productively as young mathematicians in a community of mathematical inquiry. Similarly, Lampert argues that "the knower of mathematics needs to be able to stand back from his or her own knowledge, evaluate its antecedent assumptions, argue about the foundations of its legitimacy, and be willing to have others do the same" (Lampert 1990, p. 32). As these communities of mathematical activity continue to grow through rich mathematical tasks, meaningful discourse, and ongoing reflection by both students and teachers, school mathematical practices will more closely resemble practices in the discipline of mathematics. And, like mathematicians, students will have the opportunity to develop deep understanding of important mathematical content as they are engaged in intellectually generative mathematical activity.

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ACTIVITIES

Communication; Reasoning and Proof

McGrath: "What Is the Score in Scored Discussions?"

1. *a*) Think about how you have worked in cooperative groups in the past, and score your own behavior using the rubric in figure 40.1.
   b) Critique the rubric in figure 40.1. What are the positive aspects of the rubric? The negative aspects? What are your concerns about using the rubric? Suggest any changes that you might make.

2. Write a reaction paper to this article. Include a discussion on the following statements:
   - The more students work in a group in a focused, thoughtful, and supportive way, the more they are able to learn.
   - Students need training in order to develop communication skills and engage in productive group behaviors.
   - I would, or would not, use this activity in my classroom because ....

Rubenstein: "Word Origins: Building Communication Connections"

Select three mathematical terms not listed in table 41.1. Find the root for each term, and link it with the word's mathematical meaning. Try to find at least one related word for each term. Present your findings in a format similar to table 41.1.

Peressini and Knuth: "The Role of Tasks in Developing Communities of Mathematical Inquiry"

The authors define mathematically rich tasks as those that "encourage a range of solution approaches, address significant mathematical concepts, require students to justify their explanations, and are open ended." Use the race problem and student responses to answer the following questions:

1. What is the pedagogical value in encouraging a range of solution approaches?
2. What significant mathematical concepts are involved?
3. Do the student responses show evidence of mathematical justification?
4. How, and why, would you use the race problem in the classroom?

CONNECTIONS

INSTRUCTIONAL programs from prekindergarten through grade 12 should enable all students to—

- *recognize and use connections among mathematical ideas;*
- *understand how mathematical ideas interconnect and build on one another to produce a coherent whole;*
- *recognize and apply mathematics in contexts outside of mathematics.*

The two articles illustrate how making connections can facilitate mathematical understanding. Baroody and Bartels describe how concept maps are used to develop and strengthen connections among related entities within particular mathematical topics, such as the real number system, as well as to transmit information and promote inquiry. Ensign uses written entries in student portfolios in grades K–5 to help parents and children discover and describe mathematics in their home environments and help students make connections between "school math" and the mathematics used in the "real world."

For other articles illustrating connections between mathematics and various disciplines see the following:

Social Studies
- Bay, Blesdoe, and Reys: "State-ing the Facts: Exploring the United States"
- Sawada: "Mathematics as Problem Solving: A Japanese Way"

Language Arts
- Rubenstein: "Word Origins: Building Communication Connections"

Solving Real-World Problems
- McClain, McGatha, and Hodge: "Improving Data Analysis through Discourse"
- Scavo and Petraroja: "Adventures in Statistics"

Using Concept Maps to Link Mathematical Ideas

by Athur J. Baroody and Bobbye H. Bartels

The National Council of Teachers of Mathematics's Standards documents (1989, 1991, 1995) emphasize making and assessing connections. Why are connections so important? Research suggests that understanding can be viewed as a connection between two pieces of information (Ginsburg 1977), and an understanding of elementary concepts is essential for mathematical power, for example, applying school mathematics to everyday tasks, inventing mathematical procedures, understanding and solving genuine problems, and comprehending more advanced mathematical ideas. The degree of a student's understanding is determined by the number, accuracy, and strength of connections (Hiebert and Carpenter 1992; Resnick and Ford 1981). A concept is well understood if it has many links to other aspects of knowledge that are accurate and strong.

Two examples of accurate connections follow: *(a)* addition and subtraction are inverse operations, for example, the addition of the number 3 can be undone by the subtraction of 3; and *(b)* addition facts can be used to figure out differences, for example, $5 - 3 = ?$ can be thought of as $3 + ? = 5$. An example of an inaccurate connection is that subtraction has the commutative property, for example, $5 - 3 = 3 - 5$. An example of a strong connection is consistently recognizing that all differences can be figured out by using a related addition combination. An example of a weak connection is sometimes recognizing that some subtraction combinations, such as $8 - 4$ and $12 - 6$, are related to addition doubles, such as $4 + 4 = 8$ and $6 + 6 = 12$.

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What Is a Concept Map, and Why Is It Useful?

A concept map visually illustrates mathematical connections and describes them in writing. In other words, it is a drawing and a brief description of how someone or some group thinks that certain concepts are related. Consider, for example, activity 1, which was used as a means for helping students consider the relationships within the real-number system.

Activity 1: Number system (Grade level 6–8+). Use a concept map to diagram how the following concepts are related: *(a)* composite numbers, *(b)* counting numbers, *(c)* integers, *(d)* irrational numbers, *(e)* natural numbers, *(f)* negative integers, *(g)* common fractions that do not simplify to integers, *(h)* positive integers, *(i)* primes, *(j)* rational numbers, *(k)* real numbers, *(l)* whole numbers, and *(m)* zero. Include in your diagram the following examples:

$$\frac{1}{3},\ 1,\ 2,\ -3,\ \frac{\sqrt{3}}{2},\ 6,\ 13,\ \text{and } 15.$$

The concept map illustrated in figure 44.1 summarizes how one group of preservice teachers viewed the relationships among different types of numbers. As this figure shows, a concept map consists of three elements:

1. Concept names written inside loops, rectangles, or other shapes represent concepts.
2. Linking lines, as in webbing activities or flow charts; or arrows, as in arrow diagrams, show the connections between two concepts.
3. Linking phrases, which label linking lines, describe the relationships between concepts.

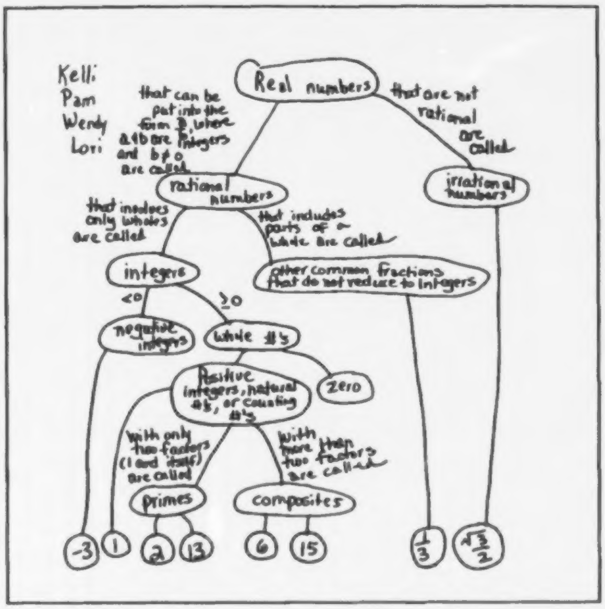
Kelli
Pam
Wendy
Lori
Real numbers
that can be put into the form $\frac{a}{b}$, where a & b are integers and b ≠ 0 are called
that are not rational are called
rational numbers
irrational numbers
that involves only wholes are called
that includes parts of a whole are called
integers
other common fractions that do not reduce to integers
<0
≥0
negative integers
whole #'s
Positive integers, natural #'s, or counting #'s
zero
with only two factors (1 and itself) are called
with more than two factors are called
primes
composites
-3
1
2
13
6
15
$\frac{1}{3}$
$\frac{\sqrt{3}}{2}$

Fig. 44.1. A hierarchical concept map of the real-number system drawn by preservice teachers

Concept maps can illustrate hierarchical relationships among ideas. In hierarchical maps, the most general concepts are placed at the top. In figure 44.1, *real numbers* is the most general concept and thus is placed above all others. More specific concepts, such as integers, whole numbers, natural numbers, and primes, are listed at successively lower levels. Such examples as 1/3, 1, 6, and 15, which are highly specific, are placed at the bottom of a hierarchical map.

Concept maps can also take the form of spider maps, which are used to illustrate webbed relationships, or chain maps, which are used to illustrate sequential relationships.

Because understanding involves "seeing" a connection and drawing a concept map requires explicitly defining connections, concept mapping can be an invaluable tool for fostering the meaningful learning of mathematics by students at any level.

What Roles Can Concept Maps Play in Meaningful Instruction?

Concept maps can be used as a teaching tool at the middle school level and beyond, including in preservice and in-service teacher education, in two different ways: *(a)* to transmit information about concepts to students or *(b)* to help students construct an understanding of concepts.

Transmitting information

A teacher could use a ready-made concept map, such as that shown in figure 44.2, as a class handout. The handout might serve as an advanced organizer, such as an overview of a lesson; a vehicle for discussing concepts and their relationships, such as a visual aid for developing a lesson; a means for tying together ideas, such as a visual aid for summarizing a lesson; or a study guide or review sheet (Bartels 1995b). In this capacity, concept maps will be beneficial in achieving a key goal of the *Curriculum and Evaluation Standards* (NCTM 1989)—helping students see connections among concepts.

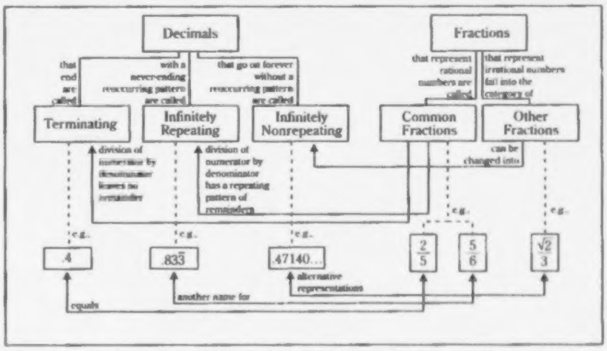
Decimals
Fractions
that end are called
with a never-ending reoccurring pattern are called
that go on forever without a reoccurring pattern are called
that represent rational numbers are called
that represent irrational numbers fall into the category of
Terminating
Infinitely Repeating
Infinitely Nonrepeating
Common Fractions
Other Fractions
division of numerator by denominator leaves no remainder
division of numerator by denominator has a repeating pattern of remainders
can be changed into
e.g.
e.g.
e.g.
e.g.
e.g.
.4
$.83\bar{3}$
.47140...
$\frac{2}{5}$
$\frac{5}{6}$
$\frac{\sqrt{2}}{3}$
alternative representations
another name for
equals

Fig. 44.2. A concept map about types of decimals and their connections to fractions, drawn by a preservice class

Constructing understanding

More consistent with the inquiry-based approach envisioned by the National Council of Teachers of Mathematics (1989, 1991), a teacher could ask students to work in groups, develop their own concept maps, and then share their results with the class. To help you better appreciate this use of concept mapping, take a moment to complete activity 2.

Activity 2: Geometric concepts (Grade level 5–8+). Using figures 1 and 2 as a guide, draw a concept map that includes the following concepts: geometry, parallelograms, plane geometry, polygons, quadrilaterals, rectangles, squares, trapezoids, and triangles. Hints: Identify the broadest concept, and place it at the top of your map. Then locate the next general concept, and place it under the first. Continue in this way until you have used all the concepts listed. To show how concepts are related, *(a)* draw in connecting lines and *(b)* label them with linking phrases that briefly explain how the concepts are related.

What Can Concept Mapping Achieve?

Concept mapping can promote inquiry-based, meaningful learning in the following ways:

1. *Provides a vehicle for introducing new concepts and connecting them to one another and to known concepts.* Asking students to include a new or unfamiliar concept in a concept map creates a need to define the term. For example, activity 1 often prompts students to ask, "What is a trapezoid?" It also creates a need to consider how trapezoids may be related to other new or previously discussed concepts, such as quadrilaterals and parallelograms.

2. *Encourages active construction of concepts.* Constructing a concept map requires explicitly illustrating and describing connections, thinking clearly and precisely about the relationships among concepts (Bartels 1995b). In other words, concept mapping prompts reflection and thus actively involves students in learning (Novak 1987). Contemplating how to draw a concept map can give students an opportunity to consider consciously how concepts are linked. To correctly place trapezoids in a concept map, for example, students must recognize that a trapezoid is a four-sided shape and thus a type of quadrilateral. To accurately label the link between quadrilaterals and trapezoids on their concept maps, students must consider how trapezoids are a special kind of quadrilateral, including what characteristics distinguish them from other quadrilaterals.

3. *Fosters metacognitive knowledge and autonomy.* In attempting to draw a concept map, students often quickly recognize for themselves what concepts and connections are not clearly or completely understood. This self-evaluation may prompt them to reason out a connection, research the topic, or ask questions. For instance, students may recognize that they have either no idea or only a fuzzy idea of what a trapezoid is.

4. *Motivates conjecture making and testing.* Deciding how to draw and label a concept can involve students in making and evaluating educated guesses. For instance, in one group, several students offered different conjectures about the definition of trapezoids: "Aren't trapezoids those things with four different-sized sides?" "Based on the picture here in our math book, a trapezoid has two parallel sides and the other two sides have to be equal." To test the latter conjecture, one student drew two pictures (and) and asked the teacher if they were both trapezoids. To the group's surprise, the teacher indicated that they were. One child concluded, "Maybe you have to have parallel sides, but the four sides could be different lengths."

5. *Underscores personal interpretation.* Typically, no one correct way of doing a concept map exists. How a map turns out depends partly on its purpose and what concepts and relationships are viewed as important and chosen for inclusion. Moreover, concept maps may reflect different interpretations of a concept. For example, because it became clear that his elementary-level students did not know the definition of a trapezoid, a teacher encouraged them to look it up. The students found that the dictionary defined trapezoid as an enclosed four-sided figure with two parallel sides. Some students interpreted this definition to mean that trapezoids had at least one pair of parallel sides; others, that trapezoids had only one pair of parallel sides. These different interpretations were reflected by the groups using different linking phrases.

6. *Provides an opportunity to engage in logical reasoning.* In considering how concepts are related, students must engage in if-then, or deductive, reasoning. For example, some students reasoned that if trapezoids had at least one pair of parallel sides, then parallelograms had to be a special kind of trapezoid and thus be connected to, and located directly under, trapezoids on their map. Others reasoned that if trapezoids had only one pair of parallel sides, then parallelograms had to be a separate class of quadrilaterals and thus not be connected to, but rather located alongside, trapezoids. The concept map illustrated in figure 44.3 is based on the second interpretation of trapezoids. Different placement of parallelograms on the concept maps offered further evidence that mathematics involves personal interpretations.

7. *Prompts problem solving.* Figuring out how to represent concepts and connections can involve students in geometric problem solving. For example, confronted with the complex task of representing how the four arithmetic operations and the properties of commutativity are interrelated, one preservice teacher proposed making a concentric hierarchical concept map by placing the broadest concept in the center and suc-

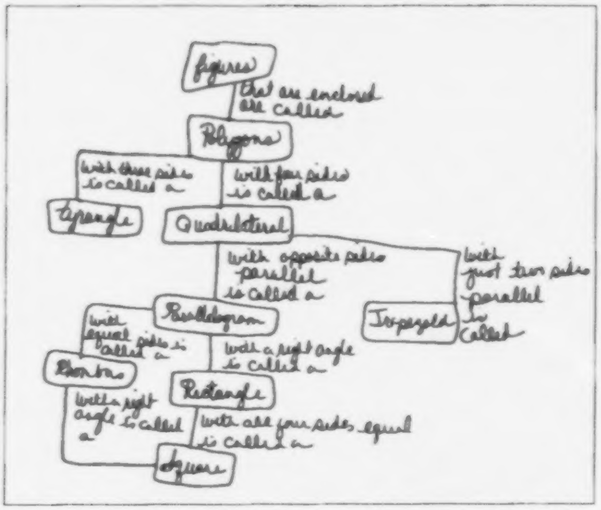
figures
that are enclosed are called
Polygons
with three sides is called a
Triangle
with four sides is called a
Quadrilateral
with opposite sides parallel is called a
with just two sides parallel is called
Parallelogram
Trapezoid
with equal sides is called a
with a right angle is called a
Rhombus
Rectangle
with a right angle is called a
with all four sides equal is called a
Square

Fig. 44.3. A fourth-fifth grader's concept map of some geometric concepts

cessively more specific concepts in successive distant rings (see fig. 44.4). This method—called "Joy's method" in honor of its creator, Joy Augustine—yielded a less confusing spatial display than the top-down hierarchical method introduced by the instructor.

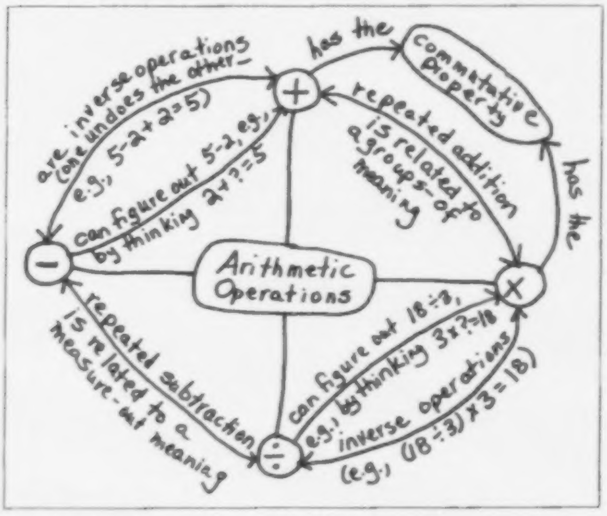
are inverse operations (one undoes the other—e.g., 5-2+2=5)
has the
Commutative Property
repeated addition is related to a groups-of meaning
has the
can figure out 5-2, e.g., by thinking 2+?=5
Arithmetic Operations
repeated subtraction is related to a measure-out meaning
can figure out 18÷3, e.g., by thinking 3×?=18
inverse operations (e.g., (18÷3)×3=18)

Fig. 44.4. Joy's (a preservice teacher's) concentric hierarchical map

8. *Kindles dialogue and a perception of the construction of mathematical knowledge as a social process.* Examining a concept map proposed by others can prompt questions, discussion, and debate. Students often differ on how a map should be constructed, and the resulting conflict can motivate map authors to defend or reevaluate their ideas (Baroody 1998). Comparing concept maps drawn for activity 1, for example, can reveal different interpretations of trapezoids and stir debate about which is correct. In this example, students may discover that either interpretation of a trapezoid given previously is workable (see, e.g., James and James [1949]) and that definitions are arbitrary, socially agreed-on conventions. Although different, multiple, or idiosyncratic interpretations sometimes make sense, at other times they are incompatible and require resolution. For example, although parallelograms might be considered a subclass of trapezoids, as some students proposed, trapezoids could not be considered a subclass of parallelograms, as others suggested. Concept mapping can provide a public forum for debating incompatible conjectures and reaching a consensus about mathematical concepts and their connections. In brief, the give-and-take prompted by sharing concept maps can underscore the fact that the construction of knowledge is very much a social process.

9. *Promotes a view of knowledge as changeable.* As children's understanding evolves, their concept maps will change. Comparing earlier and later maps on a topic can furnish concrete evidence that learning is an active, ongoing process.

10. *Creates a real need for introducing and practicing algebraic notation.* Because the space on a concept map is often limited, algebraic notation can be introduced as shorthand for writing perhaps extensive linking phrases. For instance, while representing the connections between addition and subtraction, a teacher can recommend using the connecting phrase "To do A − B = □, think B + □ = A" instead of a natural-language explanation of the complementary relationship, for example, "To do 5 − 3 ?, think what must be added to 3 to get 5." Parenthetically, concept mapping can also provide purposeful practice of such language arts skills as spelling, vocabulary, and handwriting.

Teaching Tips on Getting Started

The following tips may be helpful in getting students started with concept mapping.

- Begin with structured tasks and move gradually to less-structured tasks. Starting with an open-ended production task, which asks students to draw their own concept maps, may be too difficult for many students. Begin with a verification task, which asks students to evaluate the correctness of a completed concept map, or an add-on task, which involves adding a single concept or several concepts to an existing map. Fill-in tasks, in which students use a list of concepts and, perhaps, linking phrases to label an otherwise complete concept map; close-ended list tasks, in which students are given only a list of concepts and, perhaps, linking phrases; and then open-ended list tasks, in which students are given a list of concepts and, perhaps, linking phrases but are free to add other concepts or linking phrases, can serve as transition steps to the relatively difficult production task.

- Start with concept maps involving a few key concepts. Have students diagram just three or four main concepts in an area. Additional concepts, specific examples, and their linking phrases can be added gradually as a topic is explored at deeper levels.

- Focus first on mapping the concepts within a single topic and then on linking different maps or topics. This approach can make the construction of a relatively grand concept map manageable.

- Work together as a class to get students started on concept mapping. To introduce concept mapping, first lead a class through the evaluation, revision, or completion of a concept map several times. Students may need help learning the logic of concept maps, for example, how they can be used to represent different relations. Class discussions may also be helpful later when list tasks and other less-structured tasks are first introduced.

- Because concept mapping can be a challenging problem-solving task, many students feel more comfortable working on the maps as a team. Until they become accustomed to drawing concept maps, students will probably need supervision and feedback. With hierarchical maps, for example, students may need to be reminded about placing the most general items at the top and the most specific items, or examples, at the bottom. Remind students about labeling the connections.

- With less-structured tasks, encourage students to make a draft copy first, discuss and refine the draft, and then draw up a polished map. This process underscores the idea that concept mapping and learning itself are building processes. For example, students can see that making revisions is a natural process of any complex task. Moreover, knowing that they are working out their thoughts on a draft may be less frustrating for fastidious students.

- Encourage students to consider specific linking phrases. Concept maps are an ideal way of helping students reflect on and explicitly summarize the relationship between two concepts. However, novice concept mappers often rely too much on nondescriptive linking terms, such as *includes* or *are related* (Bartels 1995a). Although such linking terms may sometimes be correct or even ideal, they often do not spell out clearly the relationship between two concepts. For example, writing the linking phrase *with all sides equal* between rectangles and squares not only indicates an understanding that rectangles and squares are related but how they are related. If students use a nondescriptive phrase, a teacher should encourage them to be more specific.

Summary

In brief, by encouraging students to consider explicitly how concepts are linked, concept mapping can be an invaluable tool in helping them build connections and, hence, understanding. Because concept mapping requires students to portray connections explicitly, it can prompt conscious reflection on concepts, conjecturing and reasoning about the meaning of concepts and their relationships, discussion and debate about mathematical ideas, and a creative search for ways to illustrate or describe concepts and their relationships. In brief, it can be a powerful tool for fostering meaningful learning and the skills central to mathematical inquiry.

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For an illustration of how concept mapping can serve as a basis for mathematical inquiry and meaningful learning, see vignette 4 on pages 3–34 of Fostering Children's Mathematical Power: An Investigative Approach to K–8 Mathematics Instruction *(Baroody 1998) or send a request with a self-addressed, stamped envelope to Art Baroody, University of Illinois, Champaign, IL 61820.*

Parents, Portfolios, and Personal Mathematics

by Jacque Ensign

> From the standpoint of the child, the great waste in the school comes from his inability to utilize the experiences he gets outside the school in any complete and free way within the school itself; while, on the other hand, he is unable to apply in daily life what he is learning in school. That is the isolation of the school—its isolation from life. (Dewey 1905, 76)

Can we as educators consciously do something to mitigate the isolation of school from life? Can parents work with their children in recognizing and reporting on personal mathematical experiences outside of school? How can these experiences be incorporated into portfolios and into assessments of students' abilities to communicate and connect the mathematics of school and home? This article presents one attempt that was successful in connecting the in-school and out-of-school mathematical experiences of students. This attempt is an effort to tap into what homes do well and to use that success as a legitimate basis for school work. The aim is to acknowledge and celebrate families' diverse mathematical experiences outside of the classroom by incorporating them into school academics. By doing so, differences in interests, ethnicity, social class, and abilities add richness to school mathematics. The experience has challenged parents, students, and teachers to think about mathematical content in new ways and has underscored the fact that students are capable of more complexity and variety than they usually encounter in school mathematics.

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What prompted me to solicit families' experiences in mathematics was my involvement with the writing-process approach, in which students' outside experiences are often used as the basis for compositions and subsequent language arts study. Having seen the positive transformation that this approach has had on writing education, I began to wonder whether a similar approach could be used in mathematics education. I was especially concerned that students be aware of the connection between the mathematics being studied in school and the mathematics they encounter at home.

The literature agrees that out-of-school and school mathematics tend to be separated and should be linked more consistently (Resnick 1987; Nunes 1992; Apple 1992). It further suggests that out-of-school mathematics could be combined with abstract school mathematics concepts to tell a story of a particular situation in real life while showing how an abstract school mathematics concept transcends the immediate particulars of the out-of-school situation. The issue seems to be how to get the school and life experiences to work synergically so that each illuminates and reinforces the other.

Directions to the Parents

In planning what content mathematics portfolios would contain for an end-of-year assessment of each student's work, I expanded the requirements to include one section in which students were to work with their parents in describing a mathematical experience at home that related to a mathematical concept studied during the school year. I implemented this requirement for students in kindergarten through grade 5.

In a letter to parents in March, I included these directions for an addition to the child's portfolio:

> Also, the following is for your child to do and to include in the portfolio. (For those students who

> find writing difficult, it is OK for the parent to write from the student's dictation.)
>
> Write about a math problem you, the student, encountered in your everyday life this year for which you used math skills that you studied this year. Do not "make up" a problem, but rather recall one that naturally occurred at home. Describe the problem and how you solved it. Write in words (and pictures or diagrams if possible) the step-by-step process to solve this problem. Include your thoughts of how to approach the problem and your reasons.

This request required that parents know what mathematics was being studied in school to help the child recall an instance in which the year's mathematics skills were used. Parents had easy access to the ongoing portfolio of the child's work for the year. Our portfolios are bulging two-inch binders with samples of all the mathematics studied during the year. Samples of work in progress and mediocre work, as well as best work, are included in our version of portfolios. Students write, or dictate, a caption on each page to describe what was being studied in mathematics class when the page was completed. Sharing the portfolios with parents, grandparents, and friends is an integral part of communicating the mathematics being studied.

Meeting with Parent and Child

At the end of the school year, I met with each student and parent to review the student's portfolio. We spent time looking at the progression of skills throughout the year, at best work, at a piece that was a challenge for the student, and also at the everyday-mathematics problem on which the parent and child had worked together. These everyday problems brought mathematics to life for me, enabling me to peek into the world that the students experience at home. These vivid pages drew all three of us together as the student described the mathematics experience from home and how it was solved.

Reactions of Parents

As we turned to the personal-mathematics page, parents were often uneasy, wondering if they had done the assignment correctly. Some admitted that they did not understand what I wanted, saying, "We are both totally stumped by this." Some could not find any mathematics at home so had asked their child to make up a "real life" problem, saying, "I find it exceedingly difficult to think in mathematical terms." They longed for past years when I had asked to see only workbook pages.

Others said that at first they were unable to think of any mathematics experiences, but as they began to look for them, suddenly they realized that mathematics happened all the time and that they had to pick just one example. These parents had met the challenge to expand their conceptions of mathematics. One parent summarized her observation by saying, "We use math without thinking about it. When we first read your question, we thought, no, we don't do that. Then we realized we do it all the time." Another parent noted, "It's hard to think of math used in everyday life. We do it all the time but don't think of it. I guess it's because we do it all the time that we can't notice it." One parent admitted that her son thinks mathematically even though she does not: "Matthew's world revolves around math. He really sees the world through math. He saw a clock with different distinct circles. I saw just a clock." A parent of a learning-disabled student was amazed at the complexity and length of her son's solution to his problem. Indeed, I heard a number of parents express surprise and obvious pride in the sophistication of their children's solutions, far exceeding their expectations based on age or assumed ability.

Reactions of Students

Students were eager to explain what they had experienced, telling me the story behind the experience. Many expressed concern that they had not solved the problem the way that they had been taught in school to solve such problems: "I just saw the answer, I didn't have to do all those steps." Others were amazed to discover how many steps they had used: "I solved it in a way that made sense to me, but then when I had to write down how I thought it out and all the steps, that was really long."

Samples of Work

A tally of the experiences described by students in kindergarten through grade 5 includes a wide variety of practical mathematics experiences. The majority fall into the following categories.

Money

- Shopping—calculating how much can be bought, checking change, calculating price with a discount
- Mail orders—filling out order forms correctly and totaling the order
- Pets—calculating the cost to own a desired pet

- Parent's or child's business—bookkeeping, setting a price, calculating profit, handling sales transactions

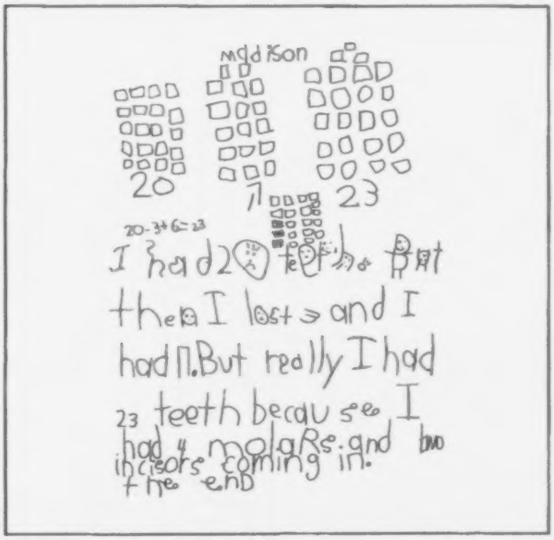
Measurement of quantity, length, and time

- Cooking—measuring ingredients, doubling recipes
- Calendar, clocks—determining how much time before an event
- Travel—calculating miles and time remaining in a car trip, cost of a trip, and miles per gallon; using maps to plot routes and calculate distances
- Crafts, models, sewing—measuring materials, following sequential directions, matching patterns

To give an idea of the variety and difficulty of problems that students tackled, I have included samples from each grade level. Noteworthy are the use of more advanced mathematics skills in home mathematics experiences than the level of mathematics that students are being asked to use in school and the complexity of those problems.

First grade

Madison (fig. 45.1) was fascinated with the number of teeth he had. His problem involved the subtraction of two-digit numbers and two further steps of addition. Not only did Madison compute with double-digit numbers (20 – 3 = 17), he then added 4 + 2 and added that result to his subtotal of 17 to equal 23. In using illustrations to help him solve his problem, he had to be very careful to keep his sets accurate and to present his problem logically.



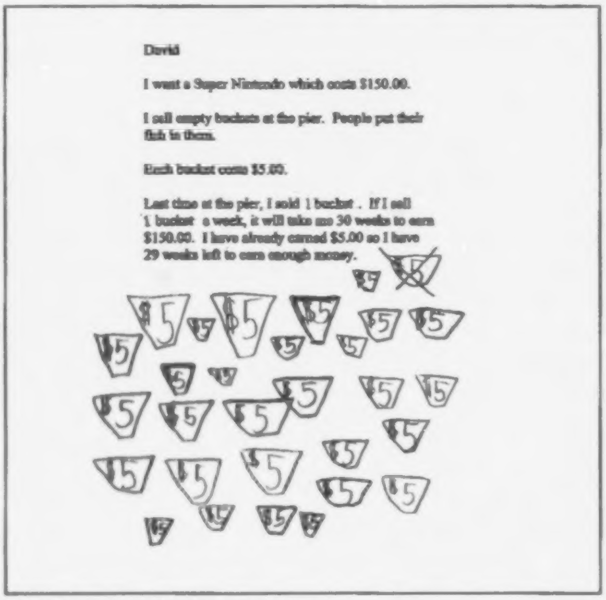
20-3+6=23

I had 20 teeth. But then I lost 3 and I had 17. But really I had 23 teeth because I had 4 molars and two incisors coming in. the end

Fig. 45.1. Madison's tooth problem

Second grade

David (fig. 45.2) was faced with the problem of not having enough money for a desired toy. Being industrious, he decided to sell fish buckets to earn his money. Solving this problem involved using repeated addition pictorially to calculate the number of weeks it would take him to earn his toy on the basis of his first week's sales record. He explained to me that he found it difficult to count the buckets from the way he had drawn them. Not only did this problem involve logic and calculations, it also challenged David's organizational skills.



David

I want a Super Nintendo which costs $150.00.

I sell empty buckets at the pier. People put their fish in them.

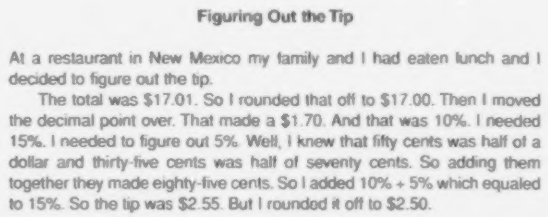
Each bucket costs $5.00.

Last time at the pier, I sold 1 bucket . If I sell 1 bucket a week, it will take me 30 weeks to earn $150.00. I have already earned $5.00 so I have 29 weeks left to earn enough money.

Fig. 45.2. David's money problem

Fourth grade

Danielle (fig. 45.3) calculated the tip for her family's meal at a restaurant. First, she rounded the total bill before calculating the tip. Her calculations involved percents of decimal numbers, which she subdivided for ease in calculating mentally. She then added her



Figuring Out the Tip

At a restaurant in New Mexico my family and I had eaten lunch and I decided to figure out the tip.

The total was $17.01. So I rounded that off to $17.00. Then I moved the decimal point over. That made a $1.70. And that was 10%. I needed 15%. I needed to figure out 5%. Well, I knew that fifty cents was half of a dollar and thirty-five cents was half of seventy cents. So adding them together they made eighty-five cents. So I added 10% + 5% which equaled to 15%. So the tip was $2.55. But I rounded it off to $2.50.

Fig. 45.3. Danielle's restaurant problem

subtotals and rounded off the answer. Rather than calculate 5 percent by multiplying, as she had for 10 percent, she chose to take half of 10 percent. Not only did she subdivide 15 percent to facilitate calculations, she also subdivided $1.70 in calculating 5 percent. If I have counted correctly, this solution has eight steps.

Fifth grade

Sean (fig. 45.4) had a very lengthy list of calculations when he decided to follow a recipe and use all of a package of cream cheese. This decision required him to multiply the entire recipe by one and a half. Had he made a large mistake on these fractions, his taco spread would not have tasted right. As it was, we enjoyed sampling his creation. Besides admiring the number of calculations that this conversion of a recipe required, I was impressed with the real-world logic that Sean used, a factor that is not included in most school problems. In calculating how many people his converted recipe would serve, he acknowledged that the answer depended on how hungry those people were.

Including Personal Mathematics in Yearly Evaluation

At the end of the school year, as part of the final assessment report for each student, I noted her or his strengths and weaknesses in meeting state standards as well as expectations based on the curriculum standards (NCTM 1989). Wanting also to stress the importance of connecting school mathematics and out-of-school mathematics, I cited the child-generated problem that had been included in the portfolio. For example, in my report on David's mathematics achievement (fig. 45.2), I wrote the following:

> An important skill in mathematics is initiating questions about problems encountered in everyday life. In solving such mathematical problems, not only are operational and computational mathematics skills from the state standards called into play, but so are the more fundamental mathematical conceptual and problem-solving abilities that the National Council of Teachers of Mathematics's Standards emphasize. For this evaluation, David was asked to write about a mathematics problem that he encountered in his everyday life this year. He was to describe the problem and his step-by-step process to solve this problem. The problem that he wrote about and illustrated was earning money to buy a Super Nintendo. He described

Layered Taco Spread

8 oz. cream cheese
8 oz. sour cream
1 jar (12 oz.) taco sauce
1 cup cheese, grated
1/2 cup chopped tomato
1/2 cup chopped green pepper
1/2 cup black olives

This is a recipe I make for my family from the book *Kids Cook!* by Sarah and Zachary Williamson. I want to make this "Layered Taco Spread," but I want to use 12 oz. of cream cheese, to use up all of the cream cheese in the refrigerator. So my problem is to figure out how much to increase all of my other ingredients.

First of all, I need to realize that 12 oz. of cream cheese is 4 oz. more than the 8 oz. called for in the recipe. 12/8 = 1 1/2, so I need 1 1/2 times the cream cheese called for in the recipe.

I now need to figure out how much of all the ingredients I need by multiplying each ingredient by 1 1/2. (1 1/2 = 3/2).

8 oz. cream cheese × 1 1/2 = 8 oz. × 3/2 = 24/2 = 12 oz. of cream cheese

8 oz. sour cream × 1 1/2 = 8 oz. × 3/2 = 24/2 = 12 oz. of sour cream

12 oz. of taco sauce × 1 1/2 = 12 oz. × 3/2 = 36/2 = 18 oz. of taco sauce

Since 8 oz. = 1 cup, I need 18/8 = 9/4 = 2 1/4 cups of sauce

1 cup of cheese × 1 1/2 = 1 × 3/2 = 3/2 = 1 1/2 cups of grated cheese

1/2 cup of chopped tomato × 1 1/2 = 1/2 × 3/2 = 3/4 = 3/4 cup of chopped tomato

1/2 cup of chopped green pepper × 1 1/2 = 1/2 × 3/2 = 3/4 = 3/4 cup of green pepper

1/2 cup of sliced black olives × 1 1/2 = 1/2 × 3/2 = 3/4 = 3/4 cup of olives

My new recipe is:

12 oz. of cream cheese
12 oz. of sour cream
18 oz., or 2 1/4 cups, of taco sauce
1 1/2 cups of grated cheese
3/4 cup of chopped tomatoes
3/4 cup of chopped green peppers
3/4 cup of black olives, sliced

Directions:

Mix the cream cheese and sour cream until smooth. A blender works great. Pour into the bottom of a pie dish. I may need 2 pie dishes now or a casserole dish.

Pour the taco sauce on top of the cream cheese mixture. Sprinkle 1/2 of the grated cheese on top.

Layer the vegetables on, one vegetable at a time.

Sprinkle with the remaining cheese on top. Serve with nacho chips.

Think of me when you make it. They say it will feed 8 people. So my new recipe should feed 12. It depends on how hungry you are.

By Sean, 5th Grade

Fig. 45.4. Sean's recipe problem

his process and graphically illustrated how many buckets and how many weeks it would take him to earn the necessary money.

Reflections and Classroom Suggestions

Rather than act only as a learning experience for the students, this assignment also proved to be the same for the parents and for me. Parents were surprised at how much mathematics their children did at home but also noted how difficult it was for them to recognize how the home mathematics related to the school mathematics. Parents had to examine carefully what their children were studying in school to help their children recognize which home mathematics experiences were related to school experiences.

I, too, learned from this experience. What impressed me the most was how much more complicated home mathematics is than the mathematics that is generally studied in school. In mathematics class, first-grade students might be doing one-step addition and subtraction problems without regrouping, whereas at home they would be solving problems that involve three-step addition and subtraction with double digits. Fourth-grade students who at school might be rounding decimals to the nearest tenth and hundredth at home might be doing eight-step calculations of percents for leaving a tip.

The richness of these out-of-school experiences in mathematics has spurred me to consider ways to allow students to share their problems with other students. Rather than ask students to relate one problem at the end of the school year, I am seeking ways to include them more frequently throughout the year. I am currently working in a different school setting, with a team of inner-city teachers and parents who use student problems in their mathematics classes. Selected student problems are discussed in class as part of lessons or reviews to enrich the mathematics experiences of all the students. Copies of student problems are sent home to parents so that they can see out-of-school mathematics that other students have described. For those students and parents who struggle with the assignments, seeing examples from other students may help them notice mathematics in their own lives.

Another strategy that we use for assisting parents is to meet with a core group of interested parents from each classroom and then to empower them to help other parents. The teachers, core parents, and I meet periodically throughout the year to share student work with parents and help them find more examples in their own homes. As students and parents become more aware of the rich mathematics experiences outside of school, I expect to see a progression of complexity and originality in the problems throughout the year. At first, rather simple examples of school mathematics being used in home situations may be submitted, but as students and parents become confident in the value of what they do at home, they may recognize more mathematics that is embedded in everyday activities. In short, I predict that students and parents will become increasingly conscious of mathematics that previously was used unconsciously in everyday life. If students' problems are celebrated and become an integral part of the school mathematics activities, I expect to see a changing understanding of the school mathematics program in the following areas:

- Students and parents will keep more abreast of what is being studied in school so as to find examples outside of school.
- School mathematics lessons will include more advanced mathematics skills and complex problems like those that students are encountering in their lives outside of school.
- Students will exhibit higher levels of understanding of mathematics as a result of the increased emphasis on finding connections between mathematics inside and outside school.

Is this assignment blurring the boundaries of language arts and mathematics, and of in-school and out-of-school activities? I hope so! In describing personal experiences, students must use not only mathematical skills but also language arts skills to communicate clearly to others the nature of the problem and the methods used to solve the problem. I see this assignment as a small step toward making school mathematics more relevant to the individual students in today's diverse world, as well as an acknowledgment of the richness of their lives outside of school.

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ACTIVITY

Connections

Baroody and Bartels: "Using Concepts Maps to Link Mathematical Ideas"

1. Select an area of mathematics, and design a sequence of three activities that would help students get started with concept mapping. Include examples of the concept maps that students might construct for each activity. See "Teaching Tips on Getting Started" for suggested activities.
2. Discuss what was difficult and easy about this activity. Discuss any mathematical insights that you gained from constructing the concept maps.

Reflecting on Principles and Standards in Elementary and Middle School Mathematics:

Readings from NCTM School-Based Journals

Edited by Anne R. Teppo

A comprehensive collection of 45 articles from *Teaching Children Mathematics*, *Mathematics Teaching in the Middle School*, the *NCTM News Bulletin*, and *Mathematics Education Dialogues*. This book of readings illustrates the various ways in which the vision of the NCTM's Principles and Standards can be actualized within the constraints, realities, and challenges of today's classrooms. It also provides helpful activities to encourage further reflection and understanding.

This publication is a companion to *Reflecting on Practice in Elementary School Mathematics* (Teppo 1999).

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